

# KLESHCHEV'S DECOMPOSITION NUMBERS FOR DIAGRAMMATIC CHEREDNIK ALGEBRAS

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**ABSTRACT.** We construct a family of graded isomorphisms between certain subquotients of diagrammatic Cherednik algebras as the quantum characteristic, multicharge, level, degree, and weighting are allowed to vary; this provides new structural information even in the case of the classical  $q$ -Schur algebra. This also allows us to prove some of the first results concerning the (graded) decomposition numbers of these algebras over fields of arbitrary characteristic.

## INTRODUCTION

Fix  $\mathbb{k}$  an algebraically closed field of characteristic  $p \geq 0$ . Given a complex reflection group of type  $G(l, 1, n)$ , a *quantum characteristic*  $e \in \{2, 3, \dots\} \cup \{\infty\}$ , and an  $e$ -*multicharge*  $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$  we have an associated cyclotomic Hecke algebra,  $\mathcal{H}_n(\kappa)$ . In the semisimple case, the simple modules of  $\mathcal{H}_n(\kappa)$  are labelled by the set of  $l$ -multipartitions,  $\mathcal{P}_n^l$ . In the non-semisimple case, Ariki's categorification theorem, [Ari96], implies that for each possible *weighting*  $\theta \in \mathbb{R}^l$  we have a corresponding parameterising set,  $\Theta \subset \mathcal{P}_n^l$ , of the simple modules of  $\mathcal{H}_n(\kappa)$ .

We wish to study these Hecke algebras via an analogue of classical Schur–Weyl duality. The appropriate language for this is provided by Rouquier's formalism of *quasi-hereditary covers* [Rou08]. In [Webb, Section 3], it is shown that the *diagrammatic Cherednik algebra*,  $A(n, \theta, \kappa)$ , is a quasi-hereditary cover of  $\mathcal{H}_n(\kappa)$  and that the simple modules of  $A(n, \theta, \kappa)$  which survive under the *Schur functor* are precisely those which are labelled by  $\Theta \subset \mathcal{P}_n^l$ . In particular the decomposition matrix of  $\mathcal{H}_n(\kappa)$  appears as a submatrix of that of  $A(n, \theta, \kappa)$ .

Over the complex field, the graded decomposition numbers of  $A(n, \theta, \kappa)$  are related to Uglov's canonical bases of higher level Fock spaces [Los16, RSVV16, Webb]. By using Uglov's construction [Ug10], one can in principle give an algorithm for computing the decomposition matrix over  $\mathbb{C}$ . However, in practice this algorithm is extremely slow. Moreover, the picture deteriorates drastically when we consider fields of prime characteristic, where almost nothing is known.

In the case that  $l = 1$  the above specialises to the study of the symmetric group and the Schur algebra of the general linear group (and their quantisations). Some of the most interesting results here have sprung from generalising Kleshchev's description of the decomposition numbers labelled by pairs of partitions which differ only by adding and removing a single node [Kle97]. This was graded and generalised to the Hecke algebra of the symmetric group by Chuang, Miyachi, Tan, and Teo (see [CMT08, TT13]) as follows. Fix  $\gamma$  a (multi)partition with no removable  $i$ -nodes and let  $\Gamma$  denote the set of all partitions which may be obtained by adding a total of  $m$   $i$ -nodes to  $\gamma$ . Given  $\lambda, \mu \in \Gamma$ , the graded decomposition number  $d_{\lambda\mu}(t)$  is given in terms of *nested sign sequences*. As well as being one of the few results which holds in positive characteristic, this result is of interest over  $\mathbb{C}$  as it provides a closed formula for  $d_{\lambda\mu}(t)$ , and so is computationally more efficient than the LLT algorithm.

In the main numerical result of this paper, we generalise the above to arbitrary diagrammatic Cherednik algebras. Over  $\mathbb{C}$ , we show that the graded decomposition numbers  $d_{\lambda\mu}(t)$  for  $\lambda, \mu \in \Gamma$  of  $A(n, \theta, \kappa)$  can be calculated in terms of nested sign sequences, see Theorem 4.12. We then show, under a mild restriction on  $\kappa$ , that the corresponding (submatrices of the) adjustment matrices for  $A(n, \theta, \kappa)$  are trivial, thereby calculating the graded decomposition numbers  $d_{\lambda\mu}(t)$  of  $A(n, \theta, \kappa)$ , for  $\lambda, \mu \in \Gamma$ , over fields of arbitrary characteristic; see Theorem 4.30.

This is done by proving a stronger, structural result over fields of arbitrary characteristic. Given  $\gamma$  a multipartition, the set  $\Gamma$  is closed under the dominance order and so there is a

strong relationship between the diagrammatic Cherednik algebra,  $A$ , and a certain subquotient  $A_{\Gamma}$ ; in particular the graded decomposition numbers and certain higher extension groups are preserved. We define a sequence  $\chi(\gamma)$  associated to a multipartition  $\gamma$ , and show that if  $\gamma$  and  $\bar{\gamma}$  are arbitrary multipartitions (which need not have the same level or degree) with  $\chi(\gamma)$  *equivalent to*  $\chi(\bar{\gamma})$ , then the corresponding subquotients  $A_{\Gamma}$  and  $A_{\bar{\Gamma}}$  are isomorphic as graded  $\mathbb{k}$ -algebras. This allows us to compare diagrammatic Cherednik algebras as the quantum characteristic, multicharge, level, degree, and weighting are all allowed to vary. This provides new structural information even in the case of the classical Schur algebras of type  $G(1, 1, n)$  (see Example 4.33) and their higher level counterparts, the cyclotomic ( $q$ -)Schur algebras of Dipper, James and Mathas [DJM98].

In [CT16] it is shown that the results of [CMT08, TT13] actually hold in more generality. As long as we never add or remove nodes whose residues differ by 1, then the graded decomposition numbers (over  $\mathbb{C}$ ) can be written as the product of the decomposition numbers for the individual residues. In this paper, we lift this result to the structural level and prove that it holds over fields of arbitrary characteristic (and generalise it to arbitrary diagrammatic Cherednik algebras) by showing that the algebras involved decompose as tensor products according to residue, see Theorem 5.4.

The paper is structured as follows. In Section 1 we recall the definition of the diagrammatic Cherednik algebra defined by Webster in [Webb] and the combinatorics underlying its representation theory. In Section 2 we recall the combinatorics of nested sign sequences from [TT13]. In Section 3 we define the subquotient algebras in which we are interested and construct cellular bases of these algebras. In Section 4 we construct a family of graded isomorphisms between the subquotient algebras. We first illustrate how one can deduce the decomposition numbers of these algebras over  $\mathbb{C}$  using only the isomorphism on the level of graded vector spaces. We then lift this to an isomorphism of graded  $\mathbb{k}$ -algebras and hence calculate the decomposition numbers over an arbitrary field  $\mathbb{k}$ . In Section 5, we then construct the isomorphism which decomposes the adjacency-free subquotient algebras as tensor products of the smaller algebras corresponding to the individual residues.

*Acknowledgements.* The authors would like to thank Joe Chuang, Anton Cox, Kai Meng Tan, and Ben Webster for helpful conversations during the preparation of this manuscript. We also thank the referee for their helpful comments. The authors are grateful for the financial support received from the Royal Commission for the Exhibition of 1851, the London Mathematical Society, and the Japan Society for the Promotion of Science.

## 1. THE DIAGRAMMATIC CHEREDNIK ALGEBRA

In this section we define the diagrammatic Cherednik algebras and recall the combinatorics underlying their representation theory.

**1.1. Graded cellular algebras with highest weight theories.** We shall study the diagrammatic Cherednik algebras through the following framework.

**Definition 1.1.** Suppose that  $A$  is a  $\mathbb{Z}$ -graded  $\mathbb{k}$ -algebra which is of finite rank over  $\mathbb{k}$ . We say that  $A$  is a *graded cellular algebra with a highest weight theory* if the following conditions hold.

The algebra is equipped with a *cell datum*  $(\Lambda, \mathcal{T}, C, \deg)$ , where  $(\Lambda, \triangleright)$  is the *weight poset*. For each  $\lambda, \mu \in \Lambda$  such that  $\lambda \triangleright \mu$ , we have a finite set, denoted  $\mathcal{T}(\lambda, \mu)$ , and we let  $\mathcal{T}(\lambda) = \cup_{\mu} \mathcal{T}(\lambda, \mu)$ . There exist maps

$$C : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \rightarrow A \quad \text{and} \quad \deg : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \rightarrow \mathbb{Z}$$

such that  $C$  is injective. We denote  $C(S, T) = c_{ST}^{\lambda}$  for  $S, T \in \mathcal{T}(\lambda)$ . We require that  $A$  satisfies properties (1)–(6), below.

- (1) Each element  $c_{S,T}^\lambda$  is homogeneous of degree

$$\deg(c_{S,T}^\lambda) = \deg(S) + \deg(T),$$

for  $\lambda \in \Lambda$  and  $S, T \in \mathcal{T}(\lambda)$ .

- (2) The set  $\{c_{S,T}^\lambda \mid S, T \in \mathcal{T}(\lambda), \lambda \in \Lambda\}$  is a  $\mathbb{k}$ -basis of  $A$ .  
 (3) If  $S, T \in \mathcal{T}(\lambda)$ , for some  $\lambda \in \Lambda$ , and  $a \in A$  then there exist scalars  $r_{S,U}(a)$ , which do not depend on  $T$ , such that

$$ac_{S,T}^\lambda = \sum_{U \in \mathcal{T}(\lambda)} r_{S,U}(a) c_{U,T}^\lambda \pmod{A^{\triangleright \lambda}},$$

where  $A^{\triangleright \lambda}$  is the  $\mathbb{k}$ -submodule of  $A$  spanned by

$$\{c_{Q,R}^\mu \mid \mu \triangleright \lambda \text{ and } Q, R \in \mathcal{T}(\mu)\}.$$

- (4) The  $\mathbb{C}$ -linear map  $*$  :  $A \rightarrow A$  determined by  $(c_{S,T}^\lambda)^* = c_{T,S}^\lambda$ , for all  $\lambda \in \Lambda$  and all  $S, T \in \mathcal{T}(\lambda)$ , is an anti-isomorphism of  $A$ .  
 (5) The identity  $1_A$  of  $A$  has a decomposition  $1_A = \sum_{\lambda \in \Lambda} 1_\lambda$  into pairwise orthogonal idempotents  $1_\lambda$ .  
 (6) For  $S \in \mathcal{T}(\lambda, \mu)$ ,  $T \in \mathcal{T}(\lambda, \nu)$ , we have that  $1_\mu c_{S,T}^\lambda 1_\nu = c_{S,T}^\lambda$ . There exists a unique element  $T^\lambda \in \mathcal{T}(\lambda, \lambda)$ , and  $c_{T^\lambda, T^\lambda}^\lambda = 1_\lambda$ .

All results in this section follow from [HM10]. Suppose that  $A$  is a graded cellular algebra with a highest weight theory. Given any  $\lambda \in \Lambda$ , the *graded standard module*  $\Delta(\lambda)$  is the graded left  $A$ -module

$$\Delta(\lambda) = \bigoplus_{\substack{\mu \in \Lambda \\ z \in \mathbb{Z}}} \Delta_\mu(\lambda)_z,$$

where  $\Delta_\mu(\lambda)_z$  is the  $\mathbb{C}$  vector-space with basis  $\{c_S^\lambda \mid S \in \mathcal{T}(\lambda, \mu) \text{ and } \deg(S) = z\}$ . The action of  $A$  on  $\Delta(\lambda)$  is given by

$$ac_S^\lambda = \sum_{U \in \mathcal{T}(\lambda)} r_{S,U}(a) c_{U,S}^\lambda,$$

where the scalars  $r_{S,U}(a)$  are the scalars appearing in condition (3) of Definition 1.1.

Suppose that  $\lambda \in \Lambda$ . There is a bilinear form  $\langle \cdot, \cdot \rangle_\lambda$  on  $\Delta(\lambda)$  which is determined by

$$c_{U,S}^\lambda c_{T,V}^\lambda \equiv \langle c_S^\lambda, c_T^\lambda \rangle_\lambda c_{U,V}^\lambda \pmod{A^{\triangleright \lambda}},$$

for any  $S, T, U, V \in \mathcal{T}(\lambda)$ . Let  $t$  be an indeterminate over  $\mathbb{Z}_{\geq 0}$ . If  $M = \bigoplus_{z \in \mathbb{Z}} M_z$  is a free graded  $\mathbb{C}$ -module, then its *graded dimension* is the Laurent polynomial

$$\text{Dim}_t(M) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{C}} M_k) t^k.$$

If  $M$  is a graded  $A$ -module and  $k \in \mathbb{Z}$ , define  $M\langle k \rangle$  to be the same module with  $(M\langle k \rangle)_i = M_{i-k}$  for all  $i \in \mathbb{Z}$ . We call this a *degree shift* by  $k$ . If  $M$  is a graded  $A$ -module and  $L$  is a graded simple module let  $[M : L\langle k \rangle]$  be the multiplicity of  $L\langle k \rangle$  as a graded composition factor of  $M$ , for  $k \in \mathbb{Z}$ .

Suppose that  $A$  is a graded cellular algebra with a highest weight theory. For every  $\lambda \in \Lambda$ , define  $L(\lambda)$  to be the quotient of the corresponding standard module  $\Delta(\lambda)$  by the radical of the bilinear form  $\langle \cdot, \cdot \rangle_\lambda$ . This module is simple, and every simple module is of the form  $L(\lambda)\langle k \rangle$  for some  $k \in \mathbb{Z}$ ,  $\lambda \in \Lambda$ . We let  $L_\mu(\lambda)$  denote the  $\mu$ -weight space  $1_\mu L(\lambda)$ . The *graded decomposition matrix* of  $A$  is the matrix  $\mathbf{D}_A(t) = (d_{\lambda\mu}(t))$ , where

$$d_{\lambda\mu}(t) = \sum_{k \in \mathbb{Z}} [\Delta(\lambda) : L(\mu)\langle k \rangle] t^k,$$

for  $\lambda, \mu \in \Lambda$ .

**Proposition 1.2** ([HM10], Proposition 2.18). *If  $\lambda, \mu \in \Lambda$  then  $\text{Dim}_t(L_\mu(\lambda)) \in \mathbb{Z}_{\geq 0}[t + t^{-1}]$ .*

**1.2. Combinatorial preliminaries.** Fix integers  $l, n \in \mathbb{Z}_{\geq 0}$ ,  $g \in \mathbb{R}_{>0}$  and  $e \in \{3, 4, \dots\} \cup \{\infty\}$ . We define a *weighting*  $\theta = (\theta_1, \dots, \theta_l) \in \mathbb{R}^l$  to be any  $l$ -tuple such that  $\theta_i - \theta_j$  is not an integer multiple of  $g$  for  $1 \leq i < j \leq l$ . Let  $\kappa$  denote an  $e$ -multicharge  $\kappa = (\kappa_1, \dots, \kappa_l) \in (\mathbb{Z}/e\mathbb{Z})^l$ .

*Remark 1.3.* We say that a weighting  $\theta \in \mathbb{R}^l$  is *well-separated* for  $A(n, \theta, \kappa)$  if  $|\theta_i - \theta_j| > ng$  for all  $1 \leq i < j \leq l$ . We say that a weighting  $\theta \in \mathbb{R}^l$  is a *FLOTW* weighting for  $A(n, \theta, \kappa)$  if  $0 < |\theta_i - \theta_j| < g$  for all  $1 \leq i < j \leq l$ .

**Definition 1.4.** An  $l$ -multipartition  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$  of  $n$  is an  $l$ -tuple of partitions such that  $|\lambda^{(1)}| + \dots + |\lambda^{(l)}| = n$ . We will denote the set of  $l$ -multipartitions of  $n$  by  $\mathcal{P}_n^l$ .

We define the *Russian array* as follows. For each  $1 \leq k \leq l$ , we place a point on the real line at  $\theta_k$  and consider the region bounded by half-lines starting at  $\theta_k$  at angles  $3\pi/4$  and  $\pi/4$ . We tile the resulting quadrant with a lattice of squares, each with diagonal of length  $2g$ .

Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)}) \in \mathcal{P}_n^l$ . The *Young diagram*  $[\lambda]$  is defined to be the set

$$\{(r, c, k) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, l\} \mid c \leq \lambda_r^{(k)}\}.$$

We refer to elements of  $[\lambda]$  as nodes (of  $[\lambda]$  or  $\lambda$ ). We define the *residue* of a node  $(r, c, k) \in [\lambda]$  to be  $\kappa_k + c - r \pmod{e}$ , and refer to  $(r, c, k)$  as an  $i$ -node if it has residue  $i$ .

We define an addable (respectively removable) node of  $\lambda$  to be any node which can be added to (respectively removed from) the diagram  $[\lambda]$  to obtain the Young diagram of a multipartition. Given  $S \subset \mathbb{Z}/e\mathbb{Z}$  we let  $\text{Rem}_S(\lambda)$  (respectively  $\text{Add}_S(\lambda)$ ) denote the set of removable (respectively addable)  $i$ -nodes of  $\lambda$  for all  $i \in S$ .

For each node of  $[\lambda]$  we draw a box in the plane; we shall draw our Young diagrams in a mirrored-Russian convention. We place the first node of component  $m$  at  $\theta_m$  on the real line, with rows going northwest from this node, and columns going northeast. The diagram is tilted ever-so-slightly in the clockwise direction so that the top vertex of the box  $(r, c, k)$  (that is, the box in the  $r$ th row and  $c$ th column of the  $k$ th component of  $[\lambda]$ ) has  $x$ -coordinate  $\theta_k + g(r - c) + (r + c)\epsilon$ .

Here the tilt  $\epsilon$  is chosen so that  $n\epsilon$  is absolutely small with respect to  $g$  (so that  $\epsilon \ll g/n$ ) and with respect to the weighting (so that  $g$  does not divide any number in the interval  $|\theta_i - \theta_j| + (-n\epsilon, +n\epsilon)$  for  $1 \leq i < j \leq l$ ).

We define a *loading*,  $\mathbf{i}$ , to be an element of  $(\mathbb{R} \times (\mathbb{Z}/e\mathbb{Z}))^n$  such that no real number occurs with multiplicity greater than one. Given a multipartition  $\lambda \in \mathcal{P}_n^l$  we have an associated loading,  $\mathbf{i}_\lambda^\theta$  (or simply  $\mathbf{i}_\lambda$  when  $\theta$  is clear from the context) given by the projection of the top vertex of each box  $(r, c, k) \in [\lambda]$  to its  $x$ -coordinate  $\mathbf{i}_{(r,c,k)} \in \mathbb{R}$ , and attaching to each point the residue  $\kappa_k + c - r \pmod{e}$  of this node. Note that the above conditions on  $\epsilon$  are designed to ensure that no two nodes have the same  $x$ -coordinate, so that  $\mathbf{i}_\lambda$  is really a loading.

We let  $D_\lambda$  denote the underlying ordered subset of  $\mathbb{R}$  given by the points of the loading. Given  $a \in D_\lambda$ , we abuse notation and let  $a$  denote the corresponding node of  $\lambda$  (that is, the node whose top vertex projects onto  $x$ -coordinate  $a \in \mathbb{R}$ ). The *residue sequence* of  $\lambda$  is given by reading the residues of the nodes of  $\lambda$  according to the ordering given by  $D_\lambda$ .

**Example 1.5.** Let  $l = 2$ ,  $g = 1$ ,  $\epsilon = 1/100$ , and  $\theta = (0, 0.5)$ . The bipartition  $((2, 1), (1^3))$  has Young diagram and corresponding loading  $\mathbf{i}_\lambda$  given in Figure 1. The residue sequence of  $\lambda$  is  $(\kappa_1 + 1, \kappa_1, \kappa_2, \kappa_1 - 1, \kappa_2 - 1, \kappa_2 - 2)$ , and the ordered set  $D_\lambda$  is  $\{-0.97, 0.02, 0.52, 1.03, 1.53, 2.54\}$ . The node  $x = -0.97$  in  $\lambda$  can be identified with the node in the first row and second column of  $\lambda^{(1)}$ .

**Definition 1.6.** Let  $\lambda, \mu \in \mathcal{P}_n^l$ . A  $\lambda$ -tableau of weight  $\mu$  is a bijective map  $T : [\lambda] \rightarrow D_\mu$  which respects residues. In other words, we fill a given node  $(r, c, k)$  of the diagram  $[\lambda]$  with a real number  $d$  from  $D_\mu$  (without multiplicities) so that the residue attached to the real number  $d$  in the loading  $\mathbf{i}_\mu$  is equal to  $\kappa_k + c - r \pmod{e}$ .

**Definition 1.7.** A  $\lambda$ -tableau,  $T$ , of shape  $\lambda$  and weight  $\mu$  is said to be semistandard if

$$\circ T(1, 1, k) > \theta_k,$$

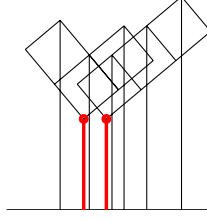


FIGURE 1. The diagram and loading of the bipartition  $((2, 1), (1^3))$  for  $l = 2$ ,  $g = 1$ ,  $\theta = (0, 0.5)$ .

- $T(r, c, k) > T(r - 1, c, k) + g$ ,
- $T(r, c, k) > T(r, c - 1, k) - g$ .

We denote the set of all semistandard tableaux of shape  $\lambda$  and weight  $\mu$  by  $\text{SStd}(\lambda, \mu)$ . Given  $T \in \text{SStd}(\lambda, \mu)$ , we write  $\text{Shape}(T) = \lambda$ .

**Example 1.8.** Fix  $\kappa = (0)$ ,  $\theta = (0)$  and  $g = 1$  and let  $\epsilon \rightarrow 0$ . For  $e = 4$ , there is a unique  $S \in \text{SStd}((3, 1), (2, 1^2))$ . This tableau is the leftmost depicted in Figure 2, below. The diagram depicts a partition of shape  $(3, 1)$  whose boxes are filled with integers. These integers are obtained from the  $x$ -coordinates of the nodes of the Young diagram  $(2, 1^2)$ . To see this, note that

$$\mathbf{i}_{(1,2,1)} = -1 + 3\epsilon \quad \mathbf{i}_{(1,1,1)} = 0 + 2\epsilon \quad \mathbf{i}_{(2,1,1)} = 1 + 3\epsilon \quad \mathbf{i}_{(3,1,1)} = 2 + 4\epsilon$$

and by letting  $\epsilon \rightarrow 0$  we obtain the entries of the tableau. One can check that this is the only tableau which satisfies the conditions in Definition 1.7. Similarly, for  $e = 5$ , there is a unique  $T \in \text{SStd}((6, 1^4), (5, 1^5))$  and a unique  $U \in \text{SStd}((6, 2^2, 1^2), (5, 3^2, 1^3))$ . These semistandard tableaux are depicted in Figure 2, below.

In all cases we let  $\epsilon \rightarrow 0$  to make the loadings easier to read.

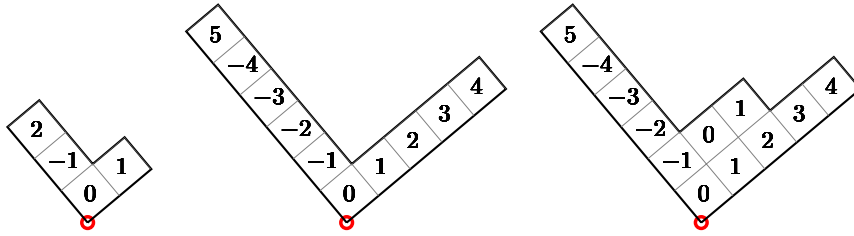


FIGURE 2. Three semistandard tableaux  $S \in \text{SStd}((3, 1), (2, 1^2))$  and  $T \in \text{SStd}((6, 1^4), (5, 1^5))$  and  $U \in \text{SStd}((6, 2^2, 1^2), (5, 2^2, 1^3))$ .

*Remark 1.9.* For many more examples of the combinatorics of loadings and tableaux, we refer the reader to [BCS, Section 2].

**Definition 1.10.** Let  $\mathbf{i}$  and  $\mathbf{j}$  denote two loadings of size  $n$  and let  $r \in \mathbb{Z}/e\mathbb{Z}$ . We say that  $\mathbf{i}$   $r$ -dominates  $\mathbf{j}$  if for every real number  $a \in \mathbb{R}$ , we have that

$$|\{(x, r) \in \mathbf{i} \mid x < a\}| \geq |\{(x, r) \in \mathbf{j} \mid x < a\}|.$$

We say that  $\mathbf{i}$  dominates  $\mathbf{j}$  if  $\mathbf{i}$   $r$ -dominates  $\mathbf{j}$  for every  $r \in \mathbb{Z}/e\mathbb{Z}$ . Given  $\lambda, \mu \in \mathcal{P}_n^l$  and  $\theta \in \mathbb{R}^l$ , we say that  $\lambda$   $\theta$ -dominates  $\mu$  (and write  $\mu \leq_\theta \lambda$ ) if  $\mathbf{i}_\lambda^\theta$  dominates  $\mathbf{i}_\mu^\theta$ .

*Remark 1.11.* We note that for  $l > 1$ , the loading of the partitions (and therefore the resulting  $\theta$ -dominance order) is heavily dependent on the weighting  $\theta \in \mathbb{R}^l$ .

**Definition 1.12.** We refer to an unordered multiset  $\mathcal{R}$  of  $n$  elements from  $(\mathbb{Z}/e\mathbb{Z})$  as a residue set of cardinality  $n$ . We let  $\mathcal{P}_n^l(\mathcal{R})$  denote the subset of  $\mathcal{P}_n^l$  whose residue set is equal to  $\mathcal{R}$ .

*Remark 1.13.* We have that  $\mathcal{P}_n^l = \cup_{\mathcal{R}} \mathcal{P}_n^l(\mathcal{R})$  is a disjoint decomposition of the set  $\mathcal{P}_n^l$ ; notice that all of the above combinatorics respects this decomposition.

*Remark 1.14.* The above combinatorics can be generalised to multi-compositions in the obvious manner.

**1.3. The diagrammatic Cherednik algebra.** Recall that we have fixed  $l, n \in \mathbb{Z}_{>0}$ ,  $g \in \mathbb{R}_{>0}$  and  $e \in \{3, 4, \dots\} \cup \{\infty\}$ . Given any weighting  $\theta = (\theta_1, \dots, \theta_l)$  and  $\kappa = (\kappa_1, \dots, \kappa_l)$  an  $e$ -multicharge, we will define what we refer to as the diagrammatic Cherednik algebra,  $A(n, \theta, \kappa)$ .

This is an example of one of many finite dimensional algebras (reduced steadied quotients of weighted KLR algebras in Webster's terminology) constructed in [Webb], whose module categories are equivalent, over the complex field, to category  $\mathcal{O}$  for a rational cyclotomic Cherednik algebra [Webb, Theorem 2.3 and 3.9]. Over fields of arbitrary characteristic and  $\theta$  a well-separated weighting, the algebra  $A(n, \theta, \kappa)$  is Morita equivalent to the corresponding cyclotomic  $q$ -Schur algebra of [DJM98] with the same level, rank, quantum characteristic and  $e$ -multicharge [Webb, Theorem 3.9].

**Definition 1.15.** We define a  $\theta$ -*diagram* of type  $G(l, 1, n)$  to be a *frame*  $\mathbb{R} \times [0, 1]$  with distinguished black points on the northern and southern boundaries given by the loadings  $\mathbf{i}_\mu$  and  $\mathbf{i}_\lambda$  for some  $\lambda, \mu \in \mathcal{P}_n^l(\mathcal{R})$  and a collection of curves each of which starts at a northern point and ends at a southern point of the same residue,  $i$  say (we refer to this as a *black  $i$ -strand*). We further require that each curve has a mapping diffeomorphically to  $[0, 1]$  via the projection to the  $y$ -axis. Each curve is allowed to carry any number of dots. We draw

- a dashed line  $g$  units to the left of each strand, which we call a *ghost  $i$ -strand* or  *$i$ -ghost*;
- vertical red lines at  $\theta_k \in \mathbb{R}$  each of which carries a residue  $\kappa_k$  for  $1 \leq k \leq l$  which we call a *red  $\kappa_k$ -strand*.

We now require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings.

*Remark 1.16.* Note that our diagrams do not distinguish between ‘over’ and ‘under’ crossings.

**Definition 1.17** ([Webb]). The *diagrammatic Cherednik algebra*,  $A(n, \theta, \kappa)$ , is the span of all  $\theta$ -diagrams modulo the following local relations (here a local relation means one that can be applied on a small region of the diagram).

- (1.1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which at no point introduces or removes any crossings of strands (black, ghost, or red).
- (1.2) For  $i \neq j$  we have that dots pass through crossings.

- (1.3) For two like-labelled strands we get an error term.

- (1.4) For double crossings of black strands, we have the following.



(1.5) If  $j \neq i - 1$ , then we can pass ghosts through black strands.

(1.6) On the other hand, in the case where  $j = i - 1$ , we have the following.

(1.7) We also have the relation below, obtained by symmetry.

(1.8) Strands can move through crossings of black strands freely.

Similarly, this holds for triple points involving ghosts, except for the following relations when  $j = i - 1$ .

(1.9)

(1.10)

In the diagrams with crossings in (1.9) and (1.10), we say that the black (respectively ghost) strand bypasses the crossing of ghost strands (respectively black strands). The ghost strands may pass through red strands freely. For  $i \neq j$ , the black  $i$ -strands may pass through red  $j$ -strands freely. If the red and black strands have the same label, a dot is added to the black strand when straightening.

(1.11)

and their mirror images. All black crossings and dots can pass through red strands, with a correction term.

(1.12)

(1.13)

(1.14)



Finally, we have the following non-local idempotent relation.

(1.15) Any idempotent where the strands can be broken into two groups separated by a blank space of size  $> g$  (so no ghost from the right-hand group can be left of a strand in the left group and *vice versa*) with all red strands in the right-hand group is referred to as unsteady and set to be equal to zero.

**1.4. The grading on the diagrammatic Cherednik algebra.** This algebra is graded as follows:

- dots have degree 2;
- the crossing of two strands has degree 0, unless they have the same label, in which case it has degree  $-2$ ;
- the crossing of a black strand with label  $i$  and a ghost has degree 1 if the ghost has label  $i - 1$  and 0 otherwise;
- the crossing of a black strand with a red strand has degree 0, unless they have the same label, in which case it has degree 1.

In other words,

$$\begin{aligned} \deg \bullet_i &= 2 & \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= -2\delta_{i,j} & \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= \delta_{j,i+1} & \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= \delta_{j,i-1} \\ & & \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= \delta_{i,j} & \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= \delta_{j,i}. \end{aligned}$$

**1.5. Representation theory of the diagrammatic Cherednik algebra.** Let  $d$  be any  $\theta$ -diagram and  $y \in [0, 1]$  be any fixed value such that there are no crossings in  $d$  at any point in  $\mathbb{R} \times \{y\}$ . Then the positions of the various strands in this horizontal slice give a loading  $\mathbf{i}_y$ . We say that the diagram  $d \in A(n, \theta, \kappa)$  *factors through* the loading  $\mathbf{i}_y$ .

The following lemma is a trivial consequence of the proof of [Webb, Lemma 2.15].

**Lemma 1.18.** *If a  $\theta$ -diagram,  $d$ , factors through a loading  $\mathbf{i}$  such that  $\mathbf{i} \triangleright \mathbf{i}_\mu$  for some  $\mu \in \mathcal{P}_n^l$ , with  $\mathbf{i}$  and  $\mathbf{i}_\mu$  not isotopic, then  $d$  factors through some  $\mathbf{i}_\lambda$  such that  $\mathbf{i}_\lambda \triangleright \mathbf{i}$ , for some  $\lambda \in \mathcal{P}_n^l$ .*

Given  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$ , we have a  $\theta$ -diagram  $B_{\mathbf{T}}$  consisting of a *frame* in which the  $n$  black strands each connecting a northern and southern distinguished point are drawn so that they trace out the bijection determined by  $\mathbf{T}$  in such a way that we use the minimal number of crossings without creating any bigons between pairs of strands or strands and ghosts. This diagram is not unique up to isotopy (since we have not specified how to resolve triple points), but we can choose one such diagram arbitrarily.

Given a pair of semistandard tableaux of the same shape  $(\mathbf{S}, \mathbf{T}) \in \text{SStd}(\lambda, \mu) \times \text{SStd}(\lambda, \nu)$ , we have a diagram  $C_{\mathbf{S}, \mathbf{T}} = B_{\mathbf{S}} B_{\mathbf{T}}^*$  where  $B_{\mathbf{T}}^*$  is the diagram obtained from  $B_{\mathbf{T}}$  by flipping it through the horizontal axis. Notice that there is a unique element  $\mathbf{T}^\lambda \in \text{SStd}(\lambda, \lambda)$  and the corresponding basis element  $C_{\mathbf{T}^\lambda, \mathbf{T}^\lambda}$  is the idempotent in which all black strands are vertical. A degree function on tableaux is defined in [Webb, Definition 2.12]; for our purposes it is enough to note that  $\deg(\mathbf{T}) = \deg(B_{\mathbf{T}})$  as we shall always work with the  $\theta$ -diagrams directly.

**Theorem 1.19** ([Webb, Section 2.6]). *The algebra  $A(n, \theta, \kappa)$  is a graded cellular algebra with a theory of highest weights. The cellular basis is given by*

$$\mathcal{C} = \{C_{\mathbf{S}, \mathbf{T}} \mid \mathbf{S} \in \text{SStd}(\lambda, \mu), \mathbf{T} \in \text{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \mathcal{P}_n^l\}$$

*with respect to the  $\theta$ -dominance order on the set  $\mathcal{P}_n^l$  and the anti-isomorphism given by flipping a diagram through the horizontal axis.*



**Theorem 1.20** ([Webb], Theorem 6.2). *Over  $\mathbb{C}$ , the (basic algebra of the) diagrammatic Cherednik algebra  $A(n, \theta, \kappa)$  is Koszul. Over  $\mathbb{C}$ , we therefore have that the graded decomposition numbers  $d_{\lambda\mu}(t) \in t\mathbb{N}_0(t)$  for  $\lambda \neq \mu \in \Lambda$ .*

*Remark 1.21.* Notice that the basis of  $A(n, \theta, \kappa)$  also respects the decomposition of  $\mathcal{P}_n^l$  by residue sets. Given a residue set  $\mathcal{R}$ , we let  $A_{\mathcal{R}}(n, \theta, \kappa)$  denote the subalgebra of  $A(n, \theta, \kappa)$  with basis given by all  $\theta$ -diagrams indexed by multipartitions  $\lambda, \mu, \nu \in \mathcal{P}_n^l(\mathcal{R})$ .

## 2. NESTED SIGN SEQUENCES

In this section we recall the combinatorics of [TT13] for calculating graded decomposition numbers. We include several illustrative examples. We fix  $i \in (\mathbb{Z}/e\mathbb{Z})$  throughout. Given  $\mu \in \mathcal{P}_n^l$ ,  $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$  and  $\theta \in \mathbb{R}^l$ , we read the loading  $\mathbf{i}_\mu$  from left to right and record any addable and removable  $i$ -nodes in order and associate the following path

$$\begin{cases} / & \text{for each removable } i\text{-node of } [\mu]; \\ \backslash & \text{for each addable } i\text{-node of } [\mu]. \end{cases}$$

Connect these line segments in order, to obtain the path  $\mathcal{P}(\mu)$ , which we refer to as the *terrain* of  $\mu$ . We place a vertex at each point where two line segments meet. If the  $j$ th edge of  $\mathcal{P}(\mu)$  is of the form  $/$  and the  $(j+1)$ th edge is of the form  $\backslash$  then we refer to the  $j$ th and  $(j+1)$ th edges as a *ridge in the terrain* of  $\mu$ . If the  $j$ th edge of  $\mathcal{P}(\mu)$  is of the form  $\backslash$  and the  $(j+1)$ th edge is of the form  $/$  then we refer to the  $j$ th and  $(j+1)$ th edges as a *valley in the terrain* of  $\mu$ .

**Example 2.1.** Let  $l = 6$  and  $n = 3$  and let  $\nu$  denote the  $l$ -multipartition  $((1), (1), \emptyset, \emptyset, \emptyset, (1))$  for  $\kappa = (1, 1, 1, 1, 1, 1)$  and some well-separated weighting  $\theta \in \mathbb{R}^6$ . The terrain of  $\mu$  is given by the leftmost diagram in Figure 3. There is a ridge between the second and third edges and a valley between the fifth and sixth edges.

Let  $l = 10$  and  $n = 6$  and let  $\mu$  denote the  $l$ -multipartition  $((1), (1), \emptyset, \emptyset, (1), \emptyset, (1), \emptyset, (1), (1))$   $\kappa = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$  and some well-separated weighting  $\theta \in \mathbb{R}^{10}$ . The terrain of  $\mu$  is given by the rightmost diagram in Figure 3.

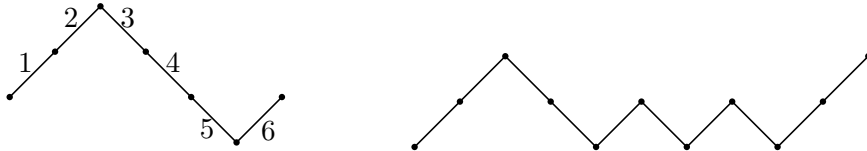


FIGURE 3. Examples of the terrain of a multipartition.

Let  $\mu, \lambda \in \mathcal{P}_n^l$  and suppose the  $\mu \triangleleft_\theta \lambda$ . We shall add a  $\lambda$ -decoration to the terrain of  $\mu$  (denoted  $\mathcal{P}(\mu, \lambda)$ ) as follows. Let  $A$  (respectively  $R$ ) denote the set of nodes in  $\lambda \setminus (\mu \cap \lambda)$  (respectively  $\mu \setminus (\mu \cap \lambda)$ ); in other words the set of nodes added to and removed from  $\mu$  to obtain the multipartition  $\lambda$ . Associate to each edge in  $A$  an opening parenthesis and to each edge in  $R$  a closing parenthesis. This defines a natural pairing on the sets  $A$  and  $R$  according to the system of nested parentheses (that these parentheses form a nesting follows from the definition of the  $\theta$ -dominance order).

We identify a pair of parentheses with the edges at which they open and close. Given  $P = (j_1, j_2), Q = (k_1, k_2) \in \mathcal{P}(\mu, \lambda)$ , we write  $P \subset Q$  if  $k_1 < j_1$  and  $j_2 < k_2$  and refer to this order as *inclusion*. We let  $\mathcal{Q}(\mu, \lambda)$  denote the partially ordered set of pairs of parentheses on  $\mathcal{P}(\mu, \lambda)$  under inclusion.

**Example 2.2.** Let  $l = 10$  and  $n = 6$  and let  $\mu = ((1), (1), \emptyset, \emptyset, (1), \emptyset, (1), \emptyset, (1), (1))$   $\lambda = ((1), (1), (1), (1), \emptyset, (1), (1), \emptyset, \emptyset, \emptyset)$  The  $\lambda$ -decorated  $\mu$ -terrain is given by the diagram in Figure 4.

FIGURE 4. The terrain of  $\mu$  decorated with multipartition  $\lambda$ .

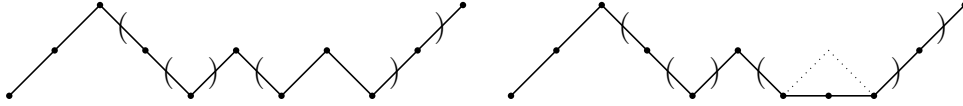
We shall now consider paths which may be obtained from  $\mathcal{P}(\mu)$  by replacing up and down edges with horizontal line segments. This requires us to slightly generalise the definition of a ridge, as follows. If the  $j$ th edge of  $\mathcal{P}(\mu)$  is of the form  $/$  and the  $k$ th edge of  $\mathcal{P}(\mu)$  is of the form  $\backslash$  and the edges strictly between  $j$  and  $k$  are all horizontal line segments, then we refer to this pair of edges as a *flattened ridge*.

**Definition 2.3.** Given  $\mu, \lambda \in \mathcal{P}_n^l$ , fix a pair of parentheses  $P \in \mathcal{Q}(\mu, \lambda)$ ; the set of *latticed paths of shape  $\mu$  and decoration  $\lambda$  with respect to the pair  $P$*  is the set of all possible ways of replacing some number of ridges formed of edges *strictly* between the parentheses to obtain flattened ridges.

We place an ordering on such paths by writing  $\rho \leq \rho'$  if the  $y$ -coordinate of every vertex in  $\rho$  is less than or equal to the  $y$ -coordinate of the corresponding vertex in  $\rho'$ .

Given  $P \in \mathcal{Q}(\mu, \lambda)$  and  $\rho$  a latticed path of shape  $\mu$  and decoration  $\lambda$ , we say that  $\rho$  has norm,  $\|\rho\|$ , given by the number of non-flattened steps strictly between the fixed pair of brackets plus 1. We refer to the unique path of maximal norm (in which no ridges are flattened) as the *generic* latticed path.

**Example 2.4.** Suppose that  $\mathcal{P}(\mu, \lambda)$  is as in Figure 4. There are no ridges strictly between the pairs of parentheses (4, 5) and so the set of latticed paths consists only of the generic path. There is a single ridge strictly between (6, 9). Therefore there are two distinct latticed paths  $\rho$  with respect to (6, 9). Namely, the generic path  $\mathcal{P}(\mu, \lambda)$  in Figure 4 and the path in which we flatten the ridge between (6, 9); these are depicted in Figure 5. These paths have norms 3 and 1, respectively.

FIGURE 5. The latticed paths of shape  $\mu$  and decoration  $\lambda$  with respect to the pair of parentheses (6, 9). These have norms 3 and 1 respectively.

**Definition 2.5.** A *well-nested latticed path* for  $\mathcal{P}(\mu, \lambda)$  is a collection  $\{\rho_P \mid P \in \mathcal{Q}(\mu, \lambda)\}$  of latticed paths such that if  $P, Q \in \mathcal{Q}(\mu, \lambda)$  and  $P \subset Q$ , then  $\rho_P \geq \rho_Q$ . We let  $\Omega(\mu, \lambda)$  denote the set of all well-nested latticed paths. The norm of a well-nested latticed path is given by the sum of the norms of the constituent paths.

**Example 2.6.** Now consider  $\mathcal{P}(\mu, \lambda)$  and the pair of parentheses given by (3, 10). There are three ridges between the pair of parentheses (3, 10). The set of all latticed paths with respect to the pair (3, 10) which are well-nested with respect to the rightmost latticed path in Figure 5 is depicted in Figure 6 below. The paths in Figure 6 are also well-nested with respect to the leftmost latticed path in Figure 5; we also obtain a further two additional paths with respect to the pair (3, 10), which are well-nested with respect to the leftmost diagram – these are depicted in Figure 7.

We have that there are a total of eight triples of latticed paths (corresponding to the three distinct pairs of parentheses), six of which are well-nested. We have that

$$\sum_{\omega \in \Omega(\mu, \lambda)} t^{\|\omega\|} = t^{11} + 2t^9 + 2t^7 + t^5.$$



FIGURE 6. The set of all latticed paths with respect to the pair  $(3, 10)$  which are well-nested with respect to the rightmost path in Figure 5. These paths have norms 5 and 3 respectively.



FIGURE 7. The additional latticed paths with respect to the pair  $(3, 10)$  which are well-nested with respect to the leftmost path in Figure 5. These paths have norms 7 and 5, respectively.

### 3. CHEREDNIK ALGEBRAS AND THEIR SUBQUOTIENTS

In this section, we shall define the subquotients of the diagrammatic Cherednik algebras in which we shall be interested for the remainder of the paper.

**Definition 3.1.** A set of residues  $S \subset \mathbb{Z}/e\mathbb{Z}$ , is said to be *adjacency-free* if  $i \in S$  implies  $i \pm 1 \notin S$ .

**Definition 3.2.** We say that  $\gamma \in \mathcal{P}_n^l(\mathcal{R})$  is *S-admissible* if  $\text{Rem}_S(\gamma) = \emptyset$ . For an *S*-admissible  $\gamma \in \mathcal{P}_n^l(\mathcal{R})$  and  $\mathcal{M}$  a multiset of *S*-residues of size  $m$ , we let  $\Gamma = \Gamma(\mathcal{M})$  denote the set of all multipartitions which may be obtained from  $\gamma$  by adding a set of nodes whose residue multiset is  $\mathcal{M}$ .

**Example 3.3.** Let  $e = 4$ ,  $g = 0.99$ ,  $\kappa = (0, 3)$ , and  $\theta = (0, 7)$ . Let  $\gamma = ((3, 2, 1^3), (4, 2^2, 1))$ ; this bipartition has residue set  $\mathcal{R} = \{0^5, 1^4, 2^5, 3^3\}$ .

Given  $S = \{1, 3\}$  and  $\gamma$  as above, we have that  $\text{Rem}_S(\gamma) = \emptyset$  and therefore  $\gamma$  is *S*-admissible. Given  $\mathcal{M} = \{1, 3^3\}$ , the set  $\Gamma = \Gamma(\mathcal{M})$  consists of the 20 bipartitions which may be obtained by adding a single 1-node and three 3-nodes to the bipartition  $\gamma$ . For example, we have that  $\alpha = ((4, 3, 1^4), (5, 2^2, 1))$  and  $\beta = ((3, 2, 1^4), (5, 2^3, 1))$  both belong to  $\Gamma$ .

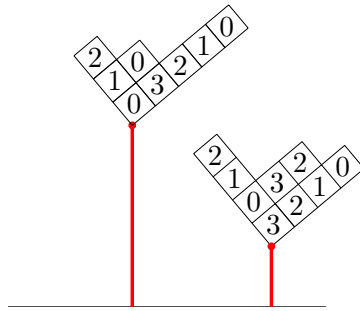


FIGURE 8. The bipartition  $\gamma = ((3, 2, 1^3), (4, 2^2, 1))$  for  $e = 4$ ,  $\kappa = (0, 3)$ ,  $g = 0.99$ , and  $\theta = (0, 7)$ .

We wish to consider the subalgebra  $A_{\mathcal{R} \cup \mathcal{M}}(n + m, \theta, \kappa)$ . In particular, we wish to consider the subquotient of  $A_{\mathcal{R} \cup \mathcal{M}}(n + m, \theta, \kappa)$  whose representations are indexed by the subset of multipartitions  $\Gamma \subset \mathcal{P}_{n+m}^l(\mathcal{R} \cup \mathcal{M})$ .

The set  $\Gamma$  has unique maximal and minimal elements under the  $\theta$ -dominance order which we shall now describe. We let  $\gamma^+$  denote the  $\triangleright_\theta$ -maximal multipartition in  $\Gamma$ ; that is, the multipartition obtained from  $\gamma$  by adding all nodes as far left as possible. Similarly, denote by  $\gamma^-$  the  $\triangleright_\theta$ -minimal multipartition in  $\Gamma$ , obtained from  $\gamma$  by adding nodes as far to the right as possible. It is clear that we can characterise  $\Gamma$  as follows:

$$\Gamma = \{\lambda \in \mathcal{P}_{n+m}^l(\mathcal{R} \cup \mathcal{M}) \mid \gamma^+ \triangleright_\theta \lambda \triangleright_\theta \gamma^-\}$$

**Example 3.4.** In the example above,  $\gamma^+ = ((4, 3, 2, 1^2), (5, 2^2, 1))$  and  $\gamma^- = ((3, 2, 1^4), (5, 2^3, 1))$ .

**Definition 3.5.** Given  $\gamma \in \mathcal{P}_n^l$ , we define idempotents

$$e = \sum_{\mu \triangleright \gamma^-} 1_\mu \quad \text{and} \quad f = \sum_{\mu \not\triangleright \gamma^+} 1_\mu$$

in  $A_{\mathcal{R} \cup \mathcal{M}}(n+m, \theta, \kappa)$ . We let  $A_\Gamma = A_\Gamma(\mathcal{M}, \theta)$  denote the subquotient of  $A_{\mathcal{R} \cup \mathcal{M}} = A_{\mathcal{R} \cup \mathcal{M}}(n+m, \theta, \kappa)$  given by

$$f(A_{\mathcal{R} \cup \mathcal{M}}/(A_{\mathcal{R} \cup \mathcal{M}} f A_{\mathcal{R} \cup \mathcal{M}}))e.$$

**Proposition 3.6.** *The algebra  $A_\Gamma$  is a graded cellular algebra with a theory of highest weights. The cellular basis is given by*

$$\{C_{S,T} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \Gamma\},$$

with respect to the  $\theta$ -dominance order on  $\Gamma$ . In particular, for  $\lambda, \mu \in \Gamma$ , we have that the graded decomposition number  $d_{\lambda\mu}(t)$  for the algebras  $A(n+m, \theta, \kappa)$  and  $A_\Gamma$  are identical, and moreover, if  $\lambda \neq \mu$ , then  $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ .

*Proof.* By definition, the sets  $E = \{\mu \mid \mu \triangleright \gamma^-\}$  and  $F = \{\mu \mid \mu \not\triangleright \gamma^+\}$  are both cosaturated (in the sense of [Don98, Appendix]) in the  $\theta$ -dominance ordering. We claim that

$$A_{\mathcal{R} \cup \mathcal{M}} f A_{\mathcal{R} \cup \mathcal{M}} = \langle C_{S,T} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu), \lambda \in F, \mu, \nu \in \mathcal{P}_n^l \rangle_{\mathbb{C}}.$$

To see that the right-hand side is contained in the left-hand side, we note that if  $S$  and  $T$  are semistandard tableaux of shape  $\lambda \in F$ , then  $C_{S,T} = B_S 1_\lambda B_T^*$  by definition. The reverse containment follows from axiom (3) of Definition 1.1 because each element  $1_\lambda = C_{T_\lambda T_\lambda}$  is itself an element of the cellular basis.

The resulting quotient algebra has basis indexed by  $S \in \text{SStd}(\lambda, \mu)$ ,  $T \in \text{SStd}(\lambda, \nu)$ ,  $\lambda, \mu, \nu \notin F$  (by conditions (2) and (3) of Definition 1.1 and Theorem 1.19). Applying the idempotent truncation to this basis (and using (6) of Definition 1.1) we obtain the required basis of  $A_\Gamma$ . The graded decomposition numbers (as well as dimensions of higher extension groups) are preserved under both the quotient and truncation maps, see for example [Don98, Appendix] for the ungraded case. Applying Theorem 1.20 will thus prove the claim about graded decomposition numbers.  $\square$

#### 4. AN ISOMORPHISM THEOREM

Let  $m, e, \bar{e} \in \mathbb{Z}_{>0}$  and let  $i \in \mathbb{Z}/e\mathbb{Z}$ ,  $\bar{i} \in \mathbb{Z}/\bar{e}\mathbb{Z}$ . Suppose  $\gamma$  and  $\bar{\gamma}$  are  $i$ -admissible and  $\bar{i}$ -admissible multipartitions, respectively. We let  $\mathcal{M}$  (respectively  $\bar{\mathcal{M}}$ ) be a set of  $m$   $i$ -nodes (respectively  $\bar{i}$ -nodes), and let  $\Gamma = \Gamma(\mathcal{M})$  and  $\bar{\Gamma} = \Gamma(\bar{\mathcal{M}})$ , i.e.  $\Gamma$  is the set of multipartitions obtained from  $\gamma$  by adding  $m$   $i$ -nodes, and similarly for  $\bar{\Gamma}$ . We shall associate a sequence  $\chi(\gamma)$  to  $\gamma$  (respectively  $\chi(\bar{\gamma})$  to  $\bar{\gamma}$ ) which records the series of  $i$ -diagonals in  $\gamma$  (respectively  $\bar{i}$ -diagonals in  $\bar{\gamma}$ ).

If  $\chi(\gamma) = \chi(\bar{\gamma})$ , then we shall show that  $A_\Gamma$  and  $A_{\bar{\Gamma}}$  are isomorphic as graded  $\mathbb{k}$ -algebras. These isomorphisms are independent of the quantum characteristic,  $e$ -multicharge, weighting, level, and degree of the corresponding diagrammatic Cherednik algebras. This provides new structural information even for the classical Schur algebras in level 1.

**4.1. Building  $i$ -diagonals from bricks.** The combinatorics needed to state and prove our isomorphism theorem is that of diagonals in the Young diagram of  $\gamma$ , which we now describe.

**Definition 4.1.** Let  $1 \leq k \leq l$  and let  $(r, c) \in [\gamma^{(k)}] \cup \text{Add}(\gamma^{(k)})$  be an  $i$ -node. We refer to the set of nodes

$$\mathbf{D} = \{(a, b) \in [\gamma^{(k)}] \mid a - b \in \{r - c - 1, r - c, r - c + 1\}\}$$

as the associated *i*-diagonal. If  $a - b$  is greater than, less than, or equal to zero, we say that the *i*-diagonal is to the left of, right of, or centred on  $\theta_k$ , respectively.

We say that an  $i$ -diagonal in  $[\gamma]$  is *visible* (respectively *invisible*) if the diagonal has (respectively does not have) an addable  $i$ -node.

**Example 4.2.** Let  $e = 5$ ,  $\gamma = (10, 9^2, 6, 4^2, 3, 2, 1^2)$ , and suppose  $\kappa = (0)$ ,  $\theta = (0)$  and  $g = 1$ . This partition contains five 0-diagonals (see Figure 9).

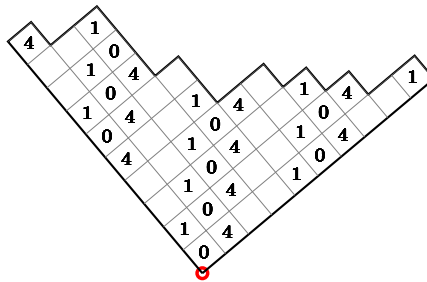


FIGURE 9. The partition  $\gamma = (10, 9^2, 6, 4^2, 3, 2, 1^2)$  with  $\kappa = (0)$  and  $e = 5$  with the diagonals of interest for  $i = 0$ . The diagram features two  $i$ -diagonals to the left of  $\theta_1$ , one centred on  $\theta_1$  and two to the right of  $\theta_1$ .

Since the multipartitions  $\lambda, \mu \in \Gamma$  differ only by moving a set of  $i$ -nodes, we are only interested in neighbourhoods (of a diagram) in which an  $i$ -strand,  $A$ , crosses a strand labelled by an  $i$ -,  $(i + 1)$ -, or  $(i - 1)$ - node in  $\gamma$ . We shall now describe all ways in which this can happen. We shall build these  $i$ -diagonals from the set of bricks  $\mathbf{B}_k$  for  $k = 1, \dots, 5$  depicted in Figure 10 and the empty brick,  $\mathbf{B}_6$ .

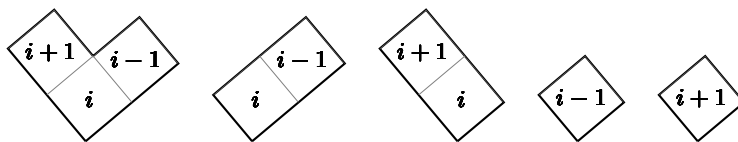


FIGURE 10. The bricks  $\mathbf{B}_1$ ,  $\mathbf{B}_2$ ,  $\mathbf{B}_3$ ,  $\mathbf{B}_4$ ,  $\mathbf{B}_5$  respectively. The  $\mathbf{B}_6$  brick is a single red  $i$ -strand (in other words it corresponds to an empty partition with charge  $i$ ).

**Case 1: Visible  $i$ -diagonals.** Fix some component  $1 \leq k \leq l$ . There are three types of  $i$ -diagonal which can occur (in this component) which have an addable  $i$ -node at the top. Namely, those which occur to the left or right of the node  $(1, 1, k)$  and those which occur on the node  $(1, 1, k)$ . It's not difficult to see that all three of these cases can be built out of the bricks  $\mathbf{B}_1$  and a single  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick respectively. Namely, we place a  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  at the base (for  $i$ -diagonals to the left, right, or centred on  $\theta_k$ , respectively) and then put some number (possibly zero) of  $\mathbf{B}_1$  bricks on top. Examples of how to construct such an  $i$ -diagonal are depicted in Figure 11.

**Case 2: Invisible  $i$ -diagonals.** Recall that we say an  $i$ -diagonal is invisible if it does not have an addable  $i$ -node at the top. Since  $\gamma$  is  $i$ -admissible, it has no removable  $i$ -nodes, and there are thus six possible invisible  $i$ -diagonals; these are obtained by adding either a  $\mathbf{B}_2$  or  $\mathbf{B}_3$  brick

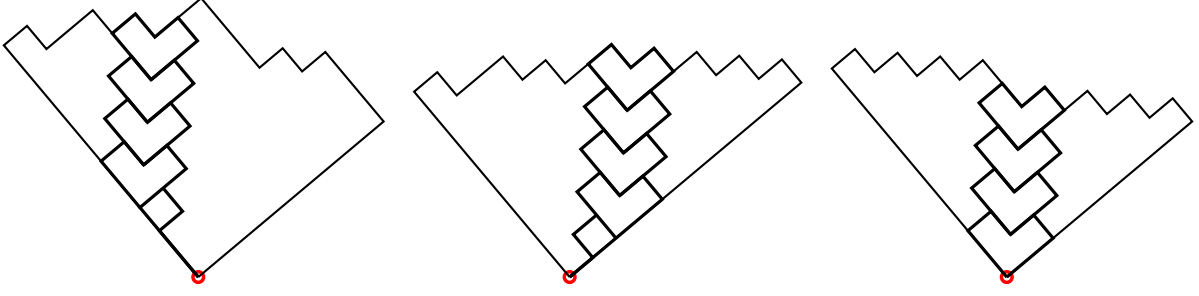


FIGURE 11. Examples of visible  $i$ -diagonals to the left, right, and centred on  $(1, 1, k)$ .

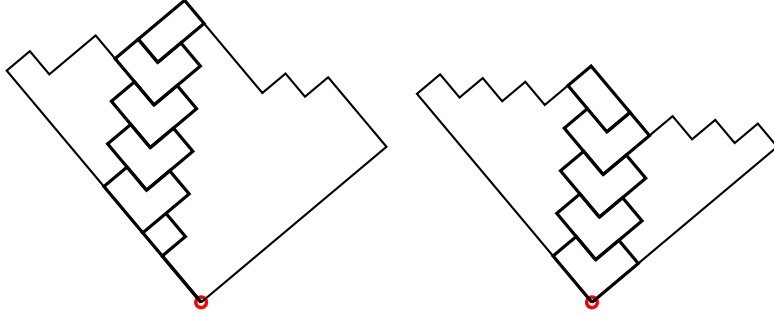


FIGURE 12. Examples of invisible  $i$ -diagonals. The former (respectively latter) is obtained by adding a  $\mathbf{B}_2$  (respectively  $\mathbf{B}_3$ ) brick to the leftmost (respectively rightmost) diagram in Figure 11.

to the top of one of the three types of visible  $i$ -diagonal. Examples of how to construct such  $i$ -diagonals are depicted in Figure 12.

Let  $\mathbf{D}$  be any  $i$ -diagonal. We define  $x(\mathbf{D})$  to be the  $x$ -coordinate of the top vertex of the top  $i$ -node in  $\mathbf{D}$  or the left vertex of the top  $(i - 1)$ -node in  $\mathbf{D}$  or the right vertex of the top  $(i + 1)$ -node in  $\mathbf{D}$  if such a node exists (if they all exist, then the definitions clearly agree).

**Example 4.3.** Continuing from Example 4.2; the ordered set of  $x$ -coordinates of the diagonals is equal to  $\{-10 + 10\epsilon, -5 + 11\epsilon, 8\epsilon, 5 + 9\epsilon, 10 + 11\epsilon\}$ .

**4.2. Strands passing through  $i$ -diagonals.** We shall let  $A$  denote an  $i$ -node of  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  and identify the node with the strand it labels in the diagram  $B_{\mathbf{T}}$ .

**Definition 4.4.** We say that an  $i$ -strand,  $A$ , *passes through* an  $i$ -diagonal,  $\mathbf{D}$ , if there is a neighbourhood of the diagram in which  $A$  is at least  $2\epsilon$  to the left of all ghost  $(i - 1)$ -strands in  $\mathbf{D}$  and a neighbourhood of the diagram in which the ghost of  $A$  is at least  $2\epsilon$  to the right of all the black  $(i + 1)$ -strands in  $\mathbf{D}$ .

*Remark 4.5.* If  $A$  satisfies Definition 4.4 then it also has the property that the ghost of  $A$  is to the left of all black  $(i + 1)$ -strands in some neighbourhood (the first in the definition) and that  $A$  is strictly to the right of all black  $i$ -strands and ghost  $(i - 1)$ -strands in some neighbourhood (the second in the definition). In this way,  $A$  passing through an  $i$ -diagonal means that  $A$  and its ghost cross all strands corresponding to the  $i$ -diagonal which may contribute to the degree or give rise to relations.

In fact, suppose that  $A$  passes through an  $i$ -diagonal  $\mathbf{D}$  and that  $\mathbf{B}$  and  $\mathbf{B}'$  are two bricks in  $\mathbf{D}$  such that  $\mathbf{B}$  is above  $\mathbf{B}'$  in the  $[\gamma]$ . Then the  $(i - 1)$ -ghost,  $i$ -strand, and  $(i + 1)$ -strand in  $\mathbf{B}$  each occur strictly to the right of the corresponding strand in  $\mathbf{B}'$ . Therefore all non-trivial interactions between  $A$  and  $\mathbf{B}$  happen before those between  $A$  and  $\mathbf{B}'$  (reading from right to left).

**Example 4.6.** Consider the diagrams  $B_{\mathbf{T}}$  and  $B_{\mathbf{U}}$  for  $\mathbf{T}$  and  $\mathbf{U}$  as in Figure 2. These are depicted in Figures 13 and 14.



Consider the diagram  $B_{\mathbf{T}}$ . This diagram has a total of three 0-diagonals;  $A$  passes through the two rightmost 0-diagonals. The first of these two crossings is with a 0-diagonal consisting of a 1-strand, a 0-strand, and a 4-strand, centred at  $\theta_1$ . The other 0-diagonal consists only of a single 4-strand (a  $\mathbf{B}_5$  brick) and is at the far right of the diagram. The total degree of the diagram is 2; the crossing of  $A$  with the centred diagonal has degree 1 (as before); the crossing of  $A$  with the rightmost diagonal also has degree 1.

Now consider the diagram  $B_{\mathbf{U}}$ . This diagram has a total of three 0-diagonals;  $A$  passes through the two rightmost 0-diagonals. The first of these two crossings is with a 0-diagonal built from a  $\mathbf{B}_7$  brick a  $\mathbf{B}_1$  brick and a  $\mathbf{B}_2$  brick. The second diagonal consists of a  $\mathbf{B}_5$  brick, as before.

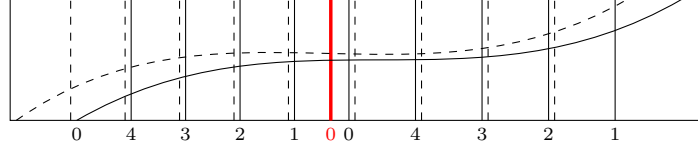


FIGURE 13. The diagram  $B_{\mathbf{T}}$  for  $\mathbf{T}$  as in Figure 2.

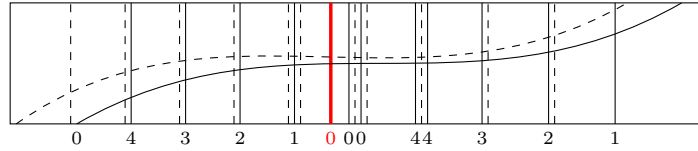


FIGURE 14. The diagram  $B_{\mathbf{U}}$  for  $\mathbf{U}$  as in Figure 2.

**4.3. A vector space isomorphism over  $\mathbb{k}$  and decomposition numbers over  $\mathbb{C}$ .** The purpose of this section is to establish the graded vector space isomorphisms between our subquotient algebras. We proceed in two steps. First, we show that for an adjacency-free residue set, we can construct a graded vector space isomorphism which allows us to address this question one-residue-at-a-time. We then construct the graded vector space isomorphisms between subquotients corresponding to a single residue. This allows us to immediately deduce the decomposition numbers of these algebras over the complex field.

Given  $\gamma$  an  $i$ -admissible multipartition, we denote the addable  $i$ -nodes of  $\gamma$  by  $A_1, A_2, \dots, A_a$  so that  $\mathbf{i}_{A_j} \leq \mathbf{i}_{A_k}$  if and only if  $j < k$ . Given  $\lambda \in \Gamma$  and  $1 \leq k \leq m$  we let  $\sigma_k(\lambda)$  denote the minimal number such that

$$|\{A_1, \dots, A_{\sigma_k(\lambda)}\} \cap [\lambda]| = k.$$

We define a length function on  $\Gamma$  as follows. Given  $\lambda, \mu \in \Gamma$  such that  $\lambda \supseteq \mu$ , we define

$$\ell(\lambda, \mu) = \sum_{1 \leq k \leq m} \sigma_k(\mu) - \sigma_k(\lambda).$$

**Example 4.7.** Let  $e = 4$  and  $\gamma, \kappa$ , and  $\theta$  be as in Example 3.3. Let  $i = 3$ ,  $\lambda = ((4, 2^2, 1^2), (5, 2^2, 1))$  and  $\mu = ((4, 2, 1^4), (4, 2^2, 1^2))$ . We have  $\sigma_1(\mu) = 1$ ,  $\sigma_2(\mu) = 4$ ,  $\sigma_3(\mu) = 5$  and  $\sigma_1(\lambda) = 1$ ,  $\sigma_2(\lambda) = 2$ ,  $\sigma_3(\lambda) = 3$  and therefore  $\ell(\lambda, \mu) = 4$ .

**Definition 4.8.** Let  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  for  $\lambda, \mu \in \Gamma$ . We define the *component word*  $R(\mathbf{T})$  of  $\mathbf{T}$  by

$$R(\mathbf{T}) = (\mathbf{T}(A_{\sigma_1(\lambda)}), \mathbf{T}(A_{\sigma_2(\lambda)}), \dots, \mathbf{T}(A_{\sigma(\lambda)})).$$

**Proposition 4.9.** Given  $\lambda, \mu \in \Gamma$ , a tableau  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  is uniquely determined by its component word  $R(\mathbf{T})$ .

*Proof.* Any node belonging to  $\gamma$  is simply mapped to itself under  $\mathbf{T}$ . Therefore a tableau  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  is an identification of  $m$  pairs of nodes  $((r, c, k), (r', c', k'))$  for  $(r, c, k) \in [\lambda \setminus \gamma]$  and  $(r', c', k') \in [\mu \setminus \gamma]$  such that  $\mathbf{i}_{(r', c', k')} \geq \mathbf{i}_{(r, c, k)}$ . It is clear that  $R(\mathbf{T})$  uniquely determines this identification of nodes in  $[\lambda \setminus \gamma]$  and  $[\mu \setminus \gamma]$  (and vice versa) and so the result follows.  $\square$

Let  $i \in \mathbb{Z}/e\mathbb{Z}$  and  $\bar{i} \in \mathbb{Z}/\bar{e}\mathbb{Z}$ . Suppose  $\gamma$  and  $\bar{\gamma}$  are  $i$ -admissible and  $\bar{i}$ -admissible multipartitions, respectively, such that  $|\text{Add}_i(\gamma)| = |\text{Add}_{\bar{i}}(\bar{\gamma})|$ . (Note that we do not assume that  $\gamma$  and  $\bar{\gamma}$  have the same level or degree.) Given  $\lambda \in \Gamma$ , we let  $\bar{\lambda} \in \bar{\Gamma}$  denote the multipartition such that  $A_{\sigma_k(\lambda)} = A_{\sigma_k(\bar{\lambda})}$  for all  $1 \leq k \leq m$ . We define a bijection  $\phi : \text{SStd}(\lambda, \mu) \rightarrow \text{SStd}(\bar{\lambda}, \bar{\mu})$  which takes  $\mathbf{T}$  to the unique  $\bar{\mathbf{T}}$  such that  $R(\mathbf{T}) = R(\bar{\mathbf{T}})$ . We let  $\Phi : A_\Gamma \mapsto A_{\bar{\Gamma}}$  denote the lift of  $\phi$  to the cellular bases of these algebras.

**Example 4.10.** Let  $e = 5$ ,  $\kappa = (0)$ ,  $g = 0.99$ ,  $\theta = (0)$ , and  $i = 0$ . The partition  $\gamma = (5, 1^4)$  is 0-admissible and  $\Gamma = \{(6, 1^4), (5, 2, 1^3), (5, 1^5)\}$ . Recall that there is a unique  $\mathbf{T} \in \text{SStd}((6, 1^4), (5, 1^5))$  of degree 2, as depicted in Figure 2.

Let  $\bar{e} = 11$ ,  $\bar{\kappa} = (1, 1, 1)$ ,  $\bar{\theta} = (-5, 0, 4)$ , and  $\bar{i} = 1$ . The multipartition  $\bar{\gamma} = (\emptyset, (2, 1), \emptyset)$  is 1-admissible and  $\bar{\Gamma} = \{(6, 1^4), (5, 2, 1^3), (5, 1^5)\}$ . There is a unique  $\mathbf{T} \in \text{SStd}(((1), (2, 1), \emptyset), (\emptyset, (2, 1), (1)))$  of degree 2.

The image under the map  $\Phi : A_\Gamma \rightarrow A_{\bar{\Gamma}}$  of the element  $B_{\mathbf{T}}$  in Figure 13 is given in Figure 15, below.

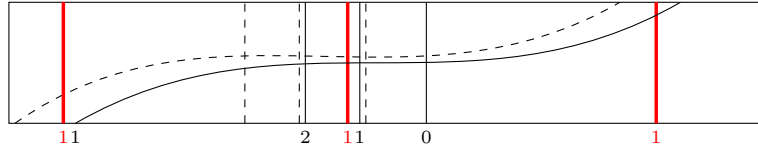


FIGURE 15. Image of the diagram in Figure 13 under  $\Phi$ , as in Example 4.10.

**Proposition 4.11.** *We have that  $A_\Gamma$  and  $A_{\bar{\Gamma}}$  are isomorphic as graded vector spaces over  $\mathbb{k}$ ; the isomorphism is given by  $\Phi(C_{\mathbf{S}, \mathbf{T}}) = C_{\bar{\mathbf{S}}, \bar{\mathbf{T}}}$ . This isomorphism preserves both the length function and the graded characters of standard modules. In other words*

$$\ell(\lambda, \mu) = \ell(\bar{\lambda}, \bar{\mu}) \quad \text{Dim}_t(\Delta_\mu(\lambda)) = \text{Dim}_t(\Delta_{\bar{\mu}}(\bar{\lambda}))$$

for all  $\lambda, \mu \in \Gamma$ .

*Proof.* We begin by explicitly describing the effect of  $\Phi$  on basis elements. Given  $(\mathbf{S}, \mathbf{T}) \in \text{SStd}(\lambda, \mu) \times \text{SStd}(\lambda, \nu)$ , the diagram of  $C_{\bar{\mathbf{S}}, \bar{\mathbf{T}}}$  may be obtained from that of  $C_{\mathbf{S}, \mathbf{T}}$  as follows.

- (1) Take the diagram corresponding to  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  and simply “forget” all black strands (and their ghosts) corresponding to nodes of  $[\gamma]$ , as well as all red strands (which are at  $x$ -coordinates given by  $\theta$ ). What remains is a diagram involving  $m$   $i$ -strands whose northern and southern points belong to the set  $\{x \mid \mathcal{D}_x \text{ is visible}\}$ .
- (2) Isotopically deform the  $i$ -strands (along with their ghosts) and their northern and southern end points (which are initially given by the loadings  $\mathbf{i}_\lambda$  and  $\mathbf{i}_\mu$  respectively) until the northern and southern end points are given by the corresponding loadings  $\mathbf{i}_{\bar{\lambda}}$  and  $\mathbf{i}_{\bar{\mu}}$  respectively. Now change the label of all of these strands from  $i$  to  $\bar{i}$ .
- (3) Finally, add the black vertical strands (and their ghosts) corresponding to nodes of  $[\bar{\gamma}]$ , as well as all red strands (which are at  $x$ -coordinates given by  $\bar{\theta}$ ).

That the map  $\Phi$  is an isomorphism of vector spaces is clear from the fact that the corresponding semistandard tableaux are in bijection.

We now wish to show that  $\Phi$  is degree preserving. Let  $(\mathbf{S}, \mathbf{T}) \in \text{SStd}(\lambda, \mu) \times \text{SStd}(\lambda, \nu)$  and let  $A$  denote any strand in the diagram  $C_{\mathbf{S}, \mathbf{T}}$  which corresponds to a removable  $i$ -node of  $\lambda$  for  $\lambda \in \Gamma$ . By assumption (and the definition of  $\Phi$ ), the strand  $A$  is common to the diagrams of both

$C_{\mathbb{S}, \tau}$  and  $C_{\mathbb{S}, \bar{\tau}}$ ; we wish to count the degree contribution of  $A$  in each case. Degree contributions are made whenever the strand  $A$  passes through an  $i$ -diagonal in  $\gamma$  (respectively  $\bar{i}$ -diagonal in  $\bar{\gamma}$ ) or an  $i$ -strand (respectively  $\bar{i}$ -strand) corresponding to a node in  $\lambda \setminus \gamma$  (respectively  $\bar{\lambda} \setminus \bar{\gamma}$ ). The strands labelled by nodes in  $\lambda \setminus \gamma$  and  $\bar{\lambda} \setminus \bar{\gamma}$  are common to both  $C_{\mathbb{S}, \tau}$  and  $C_{\mathbb{S}, \bar{\tau}}$  (with appropriate relabelling of residues), so we need only consider the degree contributions arising from  $A$  and its ghost crossing the  $i$ -diagonals in  $\gamma$  or the  $\bar{i}$ -diagonals in  $\bar{\gamma}$ .

If  $A$  passes through a  $\mathbf{B}_1$  brick then the degree contribution of this crossing is 0. If  $A$  passes through a brick  $\mathbf{B}_k$  for  $k = 4, 5, 6$  then the degree contribution of this crossing is +1. If  $A$  passes through a brick  $\mathbf{B}_k$  for  $k = 2, 3$  then the degree contribution of this crossing is -1.

Let  $\mathbf{D}$  be a diagonal in the diagram  $B_{\tau}$  and suppose that  $A$  passes through  $\mathbf{D}$ . A visible diagonal is built out of a single  $\mathbf{B}_k$  brick for  $k \in \{4, 5, 6\}$  and some number (possibly zero) of  $\mathbf{B}_1$  bricks. An invisible diagonal has an extra single  $\mathbf{B}_k$  brick for  $k \in \{2, 3\}$ . Summing over the degrees, we conclude that the crossing of  $A$  with a visible (respectively invisible)  $i$ -diagonal has degree +1 (respectively 0). Similarly for the crossing of  $A$  with a  $\bar{i}$ -diagonal in  $C_{\mathbb{S}, \bar{\tau}}$ .

That  $A_{\tau}$  and  $A_{\bar{\tau}}$  are isomorphic as graded vector spaces now follows as  $\Phi$  maps visible (respectively invisible)  $i$ -diagonals to visible (respectively invisible)  $\bar{i}$ -diagonals. That the length function is preserved is clear. That the graded dimensions of standard modules are preserved is clear from the definition of  $\Phi$  on the level of semistandard tableaux of a given shape and weight.  $\square$

We now momentarily focus our attention on the representation theory of these algebras over  $\mathbb{C}$ . In this case the diagrammatic Cherednik algebras are Koszul, and so graded decomposition numbers are particularly easy to calculate. In particular, we have the following theorem.

**Theorem 4.12.** *Let  $\gamma$  be an  $i$ -admissible multipartition. The graded decomposition numbers of  $A(n, \theta, \kappa)$  over  $\mathbb{C}$  can be given in terms of nested sign sequences as follows*

$$d_{\lambda\mu}(t) = \sum_{\omega \in \Omega(\lambda, \mu)} t^{|\omega|}$$

for  $\lambda, \mu \in \Gamma$  such that  $\mu \trianglelefteq_{\theta} \lambda$ .

*Proof.* For  $\lambda, \mu \in \Gamma$ , recall that  $\text{Dim}_t(\Delta_{\mu}(\lambda)) \in \mathbb{Z}_{\geq 0}[t, t^{-1}]$  (by the definition of the grading on basis elements),  $\text{Dim}_t(L_{\mu}(\lambda)) \in \mathbb{Z}_{\geq 0}[t + t^{-1}]$  (see Proposition 1.2), and for  $\lambda \neq \mu$  we have that  $d_{\lambda\mu}(t) \in t\mathbb{Z}_{\geq 0}[t]$  (see Theorem 1.20). It is clear that a necessary condition for the multiplicity of  $L(\mu)$  in  $\Delta(\lambda)$  to be non-zero is that  $\text{SStd}(\lambda, \mu) \neq \emptyset$ . It is also clear that  $\text{Dim}_t(\Delta_{\lambda}(\lambda)) = 1 = \text{Dim}_t(L_{\lambda}(\lambda))$ . Therefore the first five conditions of [KN10, Theorem 3.8] are satisfied, and so

$$\text{Dim}_t(\Delta_{\mu}(\lambda)) = \sum_{\substack{\nu \neq \mu \\ \text{SStd}(\nu, \mu) \neq \emptyset \\ \text{SStd}(\lambda, \nu) \neq \emptyset}} d_{\lambda\nu}(t) \text{Dim}_t(L_{\mu}(\nu)) + d_{\lambda\mu}(t).$$

Therefore, one can calculate the graded characters of simple modules and the decomposition numbers of  $A_{\Gamma}$  by induction on the distance,  $\ell(\lambda, \mu)$ , for  $\lambda, \mu \in \Gamma$  exactly as in [KN10, Main Algorithm] and [BCS, Theorem 1.18]. By Proposition 4.11, if we do this for  $\lambda, \mu \in \Gamma$  or  $\bar{\lambda}, \bar{\mu} \in \bar{\Gamma}$  we get exactly the same answer! Therefore the decomposition numbers of these algebras and the graded characters of simple modules are the same regardless of the weighting,  $e$ -multicharge, rank and level. In particular, we can run this algorithm for  $\bar{\gamma}$  a level 1 partition with  $m$  addable  $i$ -nodes. The result now follows by [TT13, Theorem 4.4] and Proposition 3.6.  $\square$

**4.4. Some useful results on moving  $i$ -strands through diagonals.** There are several sequences of relations which we will often apply in particular order during the course of the proof. For brevity, we shall now define these as Moves 1, 1\*, and 2.

**Move 1.** *Suppose we have a diagram in which two  $j$ -ghosts are not separated by a black  $(j+1)$ -strand, and the corresponding two black  $j$ -strands are separated by a  $(j-1)$ -ghost. We apply*

relation (1.10) to write this diagram as the difference of two diagrams in which the  $j$ -strands cross and the  $(j-1)$ -strand bypasses the crossing to the left or right; see Figure 16 for an example.

**Move 1\*.** Suppose we have a diagram in which two black  $j$ -strands are not separated by a  $(j-1)$ -ghost, and the corresponding two  $j$ -ghosts are separated by a black  $(j+1)$ -strand. One can repeat the above using relation (1.9) in place of (1.10).

**Move 2.** Suppose we have a diagram with a pair of crossing  $j$ -strands. We may use relations (1.3) and (1.4) to rewrite the single crossing as a double crossing with a dot on the leftmost strand located between the two crossings; see the first equality in each of Figures 17 and 18 for an example.

**Example 4.13.** Let  $e = 3$ ,  $\kappa = (2, 0)$ ,  $g = 0.99$  and  $\theta = (0, 1)$ . The leftmost diagram in Figure 16 is the idempotent corresponding to the loading of the bicomposition  $(\emptyset, (1, 2))$ . Applying Move 1 to the two adjacent 0-ghosts, we obtain the difference of two diagrams depicted in Figure 16.

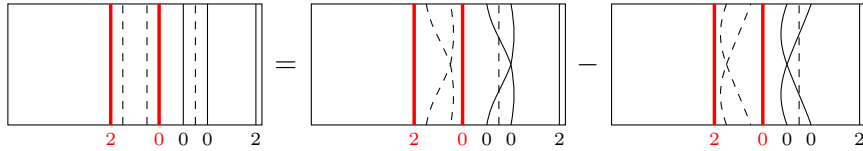


FIGURE 16. Rewriting the idempotent corresponding to the loading of  $(\emptyset, (1, 2))$  using relation (1.10).

In Figures 17 and 18 we first rewrite the right-hand side of the equality in Figure 16 using Move 2. We then use relation (1.6) (whose error term is zero by relation (1.4)) in each case to obtain an element which factors through the idempotent  $((1^2), (1))$  or an unsteady idempotent, respectively.

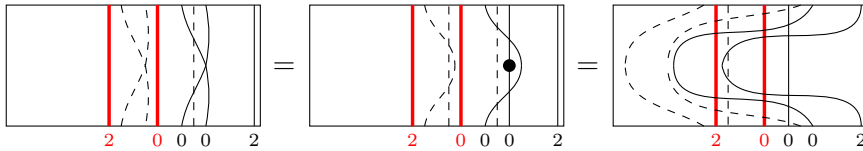


FIGURE 17. Rewriting one of the diagrams in Figure 16 as an element which factors through the idempotent labelled by  $((1^2), (1))$ .

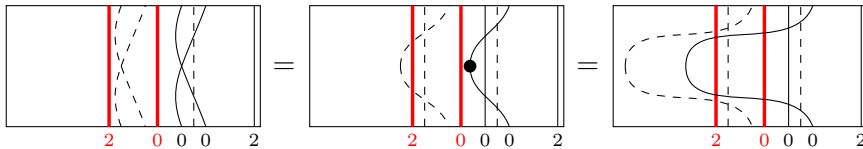


FIGURE 18. Rewriting the other diagram in Figure 16 as an element which factors through an unsteady idempotent.

The following lemmas shall also be useful in what follows.

**Lemma 4.14.** If  $d \in A(n, \theta, \kappa)$  factors through some loading  $\mathbf{i}$  such that  $\mathbf{i} \triangleright \mathbf{i}_{\gamma^+}$ , then  $d = 0$  in  $A_{\Gamma}$ . In particular, if  $d$  factors through  $1_{\lambda}$  with  $\lambda \triangleright_{(\theta, j)} \gamma^+$  for some  $j \neq i$ , then  $d = 0$  in  $A_{\Gamma}$ .

*Proof.* The result is immediate by the definition of  $A_\Gamma$ , the definition of the dominance order, and Lemma 1.18.  $\square$

**Lemma 4.15.** *Let  $A \in [\gamma]$  be at point  $x \in \mathbb{R}$  and suppose  $A \notin \text{Rem}(\gamma)$ . We let  $d \in A(n, \theta, \kappa)$  and suppose that there is a neighbourhood  $(x - n\epsilon, x + n\epsilon) \times [0, 1]$  in which*

$$d \cap ((x - n\epsilon, x + n\epsilon) \times [0, 1]) = (1_{\lambda \setminus A}) \cap ((x - n\epsilon, x + n\epsilon) \times [0, 1]),$$

*for some  $\lambda \in \Gamma$ . Then  $d = 0$  in  $A_\Gamma$ .*

*Proof.* Suppose  $A = (r, c, k)$  is a  $j$ -node. All the relations we shall apply only involve moving strands a distance less than  $n\epsilon$  to the left or right. Such relations applied to one component do not affect any other components, due to the fact that  $\epsilon$  is very small compared to the  $\theta_a - \theta_b$  for  $1 \leq a, b \leq l$ . We therefore need only focus on the interaction within the  $k$ th component.

First note that as  $(r, c, k)$  is not a removable node, there exists a node  $(r+1, c, k)$  or  $(r, c+1, k)$  in  $[\gamma]$ . We shall argue for the former case, but the latter is similar.

If  $c = 1$  and there is no node  $(r, 2, k)$  then the result follows as we need only move the nodes  $(r+a, 1, k)$  for  $a \geq 1$  to the left to obtain a loading that  $(\theta, h)$ -dominates  $\mathbf{i}_{\gamma+}$  for some  $h \neq i$ ; the result then follows by Lemma 1.18. For  $c > 1$ , we now provide an algorithm for showing that  $d = 0$  in  $A_\Gamma$ . This involves procedures on strands which we describe by the corresponding nodes in the Young diagram. If at any point in the algorithm the node to which we refer does not exist, then we have reached the first row or column of our partition; in which case terminate the algorithm and proceed to the end of the proof.

- Step 1 The  $(j-1)$ -ghosts corresponding to  $(j-1)$ -nodes  $(r+1, c, k)$  and  $(r, c-1, k)$  are not separated by a black  $j$ -strand; we can apply Move 1 to (the strands corresponding to) this pair of nodes and the  $(j-2)$ -node  $(r+1, c-1, k)$ . The result is the difference of two distinct diagrams, in which the  $(j-2)$ -strand (labelled by node  $(r+1, c-1, k)$ ) bypasses the  $(j-1)$ -crossing to the left and right. Now proceed to Step 2.
- Step 2 (a) Consider the diagram in which the  $(j-2)$ -strand bypasses to the left. Observe that the  $(j-2)$ -ghosts labelled by nodes  $(r+1, c-1, k)$  and  $(r, c-2, k)$  have no black strand separating them. The black  $(j-2)$ -strands labelled by nodes  $(r+1, c-1, k)$  and  $(r, c-2, k)$  are separated by the  $(j-3)$ -ghost strand labelled by the node  $(r+1, c-2, k)$ . We now set  $\bar{j} := j-1$  and  $(\bar{r}, \bar{c}, \bar{k}) := (r, c-1, k)$  and (using the barred residues and node labels as the input) proceed to Step 1.
- (b) Consider the diagram in which the  $(j-2)$ -strand bypasses to the right. Apply Move 2 to the crossing  $(j-1)$ -strands. Transpose the labels of the  $(j-1)$ -strands corresponding to nodes  $(r, c-1, k)$  and  $(r+1, c, k)$  (as their order when read from left to right has switched); this results in the dotted strand being labelled by  $(r, c-1, k)$ . Push the ghost of the dotted  $(j-1)$ -strand through the black  $j$ -strand immediately to its left by relation (1.6) (observe that the error term in (1.6) is zero by relation (1.4)). Observe that the  $(j-1)$ -ghosts labelled by nodes  $(r, c-1, k)$  and  $(r-1, c-2, k)$  have no black strand separating them. The black strands labelled by nodes  $(r, c-1, k)$  and  $(r-1, c-2, k)$  are separated by a  $(j-2)$ -ghost labelled by the node  $(r, c-2, k)$ . Therefore we relabel  $(\bar{r}, \bar{c}, \bar{k}) := (r-1, c-1, k)$  and  $\bar{j} := j$  and proceed to Step 1.

The algorithm terminates if at the end of Step 2 case (a) we set  $\bar{c} = 0$  and in case (b) we set  $\bar{r} = 0$  or  $\bar{c} = 0$ . If we terminate in case (a), then our diagram has a crossing pair of black  $(\bar{j}-1)$ -nodes labelled by  $(\bar{r}, 1)$  and  $(\bar{r}+1, 2)$  bypassed by a ghost  $(\bar{j}-2)$ -strand to the left. We can pull the  $(\bar{j}-2)$ -strand at least  $n\epsilon$  units to the left; we then apply Move 2 to the crossing  $(\bar{j}-1)$ -strands and pull the ghost dotted  $(\bar{j}-1)$ -strand through the black  $\bar{j}$ -strand immediately to its left. The loading at  $y = 1/2$  in the resulting diagram  $(\theta, h)$ -dominates  $\mathbf{i}_{\gamma+}$  for  $h = \bar{j}-1, \bar{j}-2$ . The result follows from Lemma 4.14. Case (b) is similar.  $\square$

**Lemma 4.16.** *Given  $\lambda \in \Gamma$ , if we add a dot to any of the strands in  $1_\lambda$  corresponding to a node in  $\gamma$ , then the resulting diagram is zero in  $A_\Gamma$ .*

*Proof.* Let  $A = (r, c, k)$  denote a  $j$ -node in  $\gamma$  with a dot on the corresponding strand. We proceed by induction on  $r + c$ ; in the case  $(r, c, k) = (1, 1, k)$ ,  $A$  can pass through the red  $j$ -strand immediately to its left using relation (1.11). If  $j \neq i$  then the diagram is zero by Lemma 4.14. If  $j = i$ , then our assumption that  $(r, c, k) \in \gamma$  for  $\gamma$   $i$ -admissible implies that there is either an  $(i - 1)$ -node  $(2, 1, k)$  or an  $(i + 1)$ -node  $(1, 2, k)$ . In either case, the diagram is zero by Lemma 4.14.

We now assume that  $r + c \geq 2$ . We can pull  $A$  through the  $(j + 1)$ -ghost to its left, labelled by  $(r - 1, c, k)$ , at the expense of losing the dot (we also obtain an error term  $1_\lambda$  with a dot on the  $(j + 1)$ -strand labelled by  $(r - 1, c, k)$ , which is zero by induction). We now apply Move 1 to the  $j$ -ghosts labelled by nodes  $(r, c, k)$  and  $(r - 1, c - 1, k)$  and the  $(j + 1)$ -strand labelled by  $(r - 1, c, k)$ , to obtain two terms. The term in which the  $(j + 1)$ -strand bypasses the crossing of  $j$ -ghosts to the left is zero by Lemma 4.15.

Now consider the remaining term in which the  $(j + 1)$ -strand bypasses the crossing of  $j$ -ghosts to the right. If  $j \neq i$ , the result follows by Lemma 4.14. If  $j = i$ , we continue by applying Move 2 and pulling the dotted strand to the left. Repeating this argument we can pull  $A$  through all the  $i$ -strands and onwards outside of the region in Lemma 4.15 and the result follows.  $\square$

We denote the  $i$ -diagonals in  $\gamma$  by  $\mathbf{D}_{x_1}, \mathbf{D}_{x_2}, \dots$  so that  $x_a = x(\mathbf{D}_{x_a})$  and  $x_a < x_b$  whenever  $a < b$ . We let  $b_a := b_a(\mathbf{D})$  denote the total number of  $\mathbf{B}_a$  bricks in the  $i$ -diagonal  $\mathbf{D}$  for  $a = 1, \dots, 6$ .

**Proposition 4.17.** *We can pull an  $i$ -crossing through an  $i$ -diagonal  $\mathbf{D}$  at the expense of an error term, as illustrated in Figure 19.*

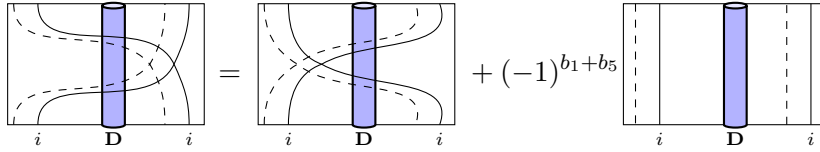


FIGURE 19. Pulling an  $i$ -crossing through an  $i$ -diagonal  $\mathbf{D}$ . Recall,  $b_k := b_k(\mathbf{D})$  is the total number of  $\mathbf{B}_k$  bricks in  $\mathbf{D}$ .

We shall prove the proposition via a series of small lemmas representing easy cases. Recalling Remark 4.5, we proceed from right-to-left through the possible bricks that form an  $i$ -diagonal, and check what happens as the  $i$ -crossing passes each successive brick.

If the  $i$ -diagonal is invisible, we first must pass the  $i$ -crossing through either a  $\mathbf{B}_3$  or  $\mathbf{B}_2$  brick. We shall show that the  $i$ -crossing passes through this brick without cost.

**Lemma 4.18.** *We can pull an  $i$ -crossing through a  $\mathbf{B}_2$  or  $\mathbf{B}_3$  brick without cost.*

*Proof.* In the former (respectively latter) case, we first apply relation (1.9) (respectively (1.10)) to the ghost  $i$ -crossing and the black  $(i + 1)$ -strand (respectively  $i$ -crossing and the  $(i - 1)$ -ghost) to push the  $i$ -crossing through to the left at the expense of an error term. In both cases, the error term is zero by relation (1.4). The  $\mathbf{B}_2$  case is illustrated in Figure 20. We may now pull the  $i$ -crossing through the black  $i$ -strand without cost (by relation (1.8)). We therefore obtain the required diagram.  $\square$

We have seen that we can pull an  $i$ -crossing pair through a  $\mathbf{B}_2$  or  $\mathbf{B}_3$  brick without cost. Therefore, we now consider what happens when we pull an  $i$ -crossing through some number (possibly zero) of  $\mathbf{B}_1$  bricks. We first deal with the case that  $b_1 = 0$ .

**Lemma 4.19.** *Let  $\mathbf{D}$  be an  $i$ -diagonal with  $b_1 = 0$ . We can pull an  $i$ -crossing through  $\mathbf{D}$  at the expense of an error term, as illustrated in Figure 19.*



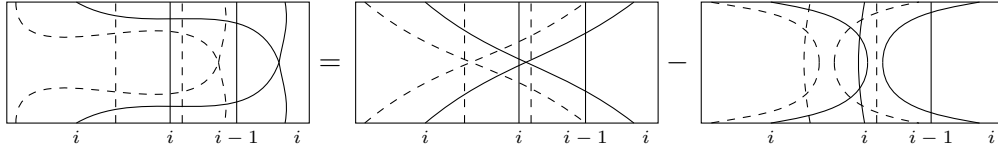


FIGURE 20. Pulling an  $i$ -crossing through a  $\mathbf{B}_2$  brick. On the right-hand side of the equality the first diagram can now be pulled to the left through the  $i$ -strand at no cost. The second diagram is zero by relation (1.4).

*Proof.* By Lemma 4.18 we need only consider pulling an  $i$ -crossing through a  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick. This can be done at the expense of an error term as in relations (1.9), (1.10) and (1.12), giving the required form.  $\square$

**Lemma 4.20.** *Let  $\mathbf{D}$  be an  $i$ -diagonal with  $b_1 = 1$ . We can pull an  $i$ -crossing through  $\mathbf{D}$  at the expense of an error term, as illustrated in Figure 19.*

*Proof.* As in Lemma 4.19, we need only consider pulling an  $i$ -crossing through a  $\mathbf{B}_1$  brick followed by a  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick. We shall prove this via a series of steps.

**Step 1.** We first pull the  $i$ -crossing through the  $(i-1)$ -ghost at the expense of an error term (in which we undo the crossing) using relation (1.10). The error term is the leftmost diagram depicted in Figure 21.

**Step 2.** We can now apply relation (1.9) to pass the ghost  $i$ -crossing through the  $(i+1)$ -strand and obtain a further error term. This error term is easily seen to be zero by applying relation (1.4). This gives us the first term after the equality in Figure 22.

**Step 3.** We now turn our attention to the error term from Step 1. We pull the non-vertical  $i$ -ghost through the vertical black  $(i+1)$ -strand immediately to its left. The result is a diagram with a double crossing of black  $i$ -strands with a dot on the rightmost of the two, this is depicted in Figure 21. We also obtain an error term with a dot on the  $(i+1)$ -strand; however this error term is zero by relation (1.4) and so is not depicted in Figure 21.

**Step 4.** Continuing from Step 3, we can apply relations (1.3) and (1.4) to rewrite the dotted double  $i$ -crossing as a single crossing without decoration at the expense of multiplication by the scalar  $-1$ . This diagram can then be deformed isotopically to obtain the rightmost diagram in Figure 22.

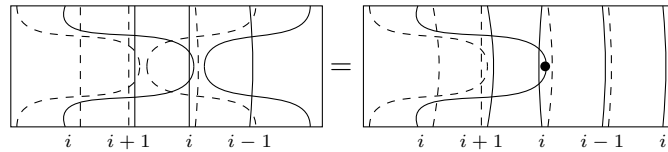


FIGURE 21. The diagram on the left-hand side of the equality is the error term from Step 1. The diagram on the right-hand side is obtained by applying relation (1.7); the resulting error term is then zero by relation (1.4).

Applying steps 1–4 pulls the  $i$ -crossing through the  $\mathbf{B}_1$  brick as depicted in Figure 22. Finally, we may pull the  $i$ -crossing through the  $\mathbf{B}_4$ ,  $\mathbf{B}_5$  or  $\mathbf{B}_6$  brick. Doing so for the first term after the equality in Figure 22 yields the first term in Figure 19 and an error term which is zero by relation (1.4). Doing so for the second term after the equality in Figure 22 yields a term which is zero by Lemma 4.15 and an error term which is the second term in Figure 19.  $\square$

We now turn our attention to proving Proposition 4.17 in full generality. We refer to the rightmost diagram in Figure 22 as having *an  $i$ -crossing attached to the  $i$ -node in the  $\mathbf{B}_1$  brick*.

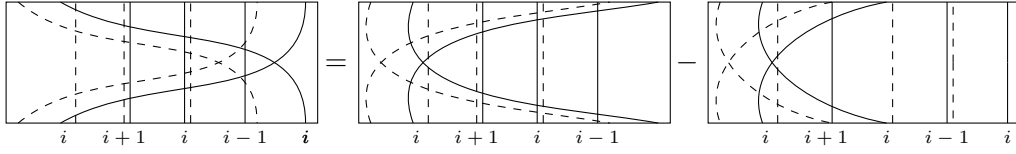


FIGURE 22. Pulling an  $i$ -crossing through a single  $\mathbf{B}_1$  brick. We have pulled the  $i$ -crossing as far to the left as possible in order to illustrate that it has passed through the brick; however, in practice it will have to pass through other bricks.

*Proof of Proposition 4.17.* By Lemmas 4.18 to 4.20 we may assume that we start by passing the  $i$ -crossing through  $b_1$   $\mathbf{B}_1$  bricks for  $b_1 \geq 2$ .

Repeating the first step in the argument in Lemma 4.20 yields a leading term and an error term. By repeatedly applying relations (1.9) and (1.10), we can push the  $i$ -crossing in the leading term through all  $b_1$   $\mathbf{B}_1$  bricks; each error term along the way is zero by relation (1.4). We can then proceed to push the  $i$ -crossing through the  $\mathbf{B}_4$ ,  $\mathbf{B}_5$  or  $\mathbf{B}_6$  brick, yielding the first term in Figure 19 and an error term which is again zero by relation (1.4).

We now deal with the error term from our first step. As in Steps 3 and 4 of the proof of Lemma 4.20, we can rewrite this as  $-1$  multiplied by the diagram with a crossing attached to the  $i$ -node (say  $(r, c, k)$ ) in the top  $\mathbf{B}_1$  brick.

We now diverge from the proof of Lemma 4.20, as we need to consider what happens when we pull the  $i$ -crossing attached to  $(r, c, k)$  to the left. Firstly, we must pull this  $i$ -crossing through the next  $\mathbf{B}_1$  brick. We pull the  $i$ -crossing through the  $(i-1)$ -ghost labelled by the node  $(r, c-1, k)$  yielding two terms: the leading term  $d$  and an error term  $d'$ . The term  $d$  is zero, as we can now push this  $i$ -crossing through all remaining  $\mathbf{B}_1$  bricks and the  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick and apply Lemma 4.15, with all error terms along the way being zero by relation (1.4). Now, observe that the diagram  $d'$  has a double crossing of  $i$ -strands. We can apply relation (1.6) to  $d'$ , followed by relations (1.3) and (1.4) to rewrite the double crossing as an  $i$ -crossing attached to the node  $(r-1, c-1, k)$ , at the expense of scalar multiplication by  $-1$  again.

We repeat the above procedure until we end up with a diagram with an  $i$ -crossing attached to node  $(r', c', k)$  with  $r' = 1$  or  $c' = 1$  (that is, attached to the  $i$ -node of the bottom  $\mathbf{B}_1$  brick) and coefficient  $(-1)^{b_1}$ . Finally, we pull this  $i$ -crossing through the  $\mathbf{B}_4$ ,  $\mathbf{B}_5$  or  $\mathbf{B}_6$  brick to yield a leading term which is zero by Lemma 4.15 and an error term which is the second term in Figure 19.  $\square$

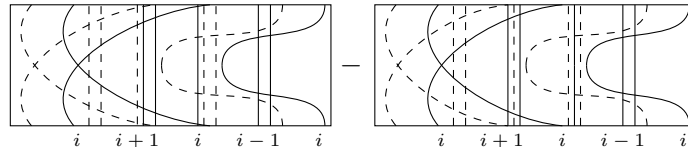
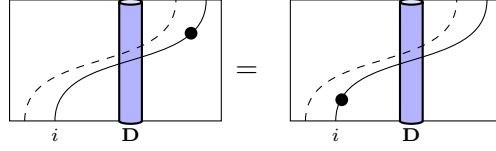


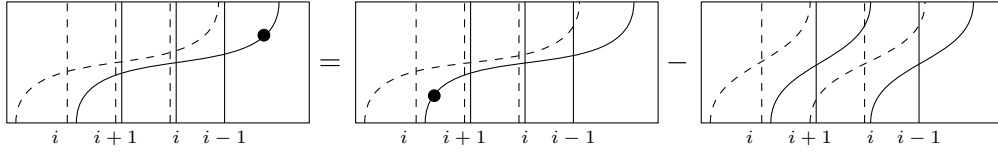
FIGURE 23. Pulling the  $i$ -crossing in the rightmost diagram in Figure 22 through a second  $\mathbf{B}_1$  brick. The leftmost diagram becomes zero once we push the  $i$ -crossing through all bricks, by Lemma 4.15.

**Proposition 4.21.** *We can pull a dot through an  $i$ -diagonal,  $\mathbf{D}$ , without cost, as illustrated in Figure 24.*

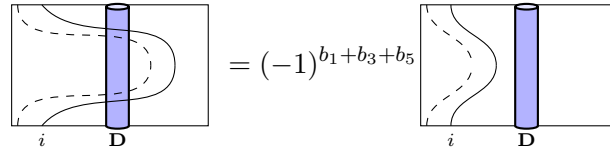
*Proof.* The result is clear for bricks of the form  $\mathbf{B}_4$ ,  $\mathbf{B}_5$  or  $\mathbf{B}_6$  by applying relations (1.2) and (1.14). Now assume that  $\mathbf{D}$  has more than one brick. Each brick of the form  $\mathbf{B}_1$ ,  $\mathbf{B}_2$ ,  $\mathbf{B}_3$  contains a single  $i$ -strand. This strand is intersected by the  $i$ -strand in Figure 24. We can pull the dot

FIGURE 24. Pulling a dot through an  $i$ -diagonal.

through the brick (using relation (1.3)) at the expense of an error term in which we undo the aforementioned  $i$ -crossing. The resulting diagram factors through an idempotent which is zero by Lemma 4.15. See Figure 25 for the case of a single  $\mathbf{B}_1$  brick.  $\square$

FIGURE 25. Pulling a dot through a  $\mathbf{B}_1$  brick. The rightmost diagram is zero modulo in  $A_\Gamma$  by Lemma 4.15.

**Proposition 4.22.** *Suppose we have a double crossing of an  $i$ -strand,  $A$ , with an invisible  $i$ -diagonal,  $\mathbf{D}$ . We can pull  $A$  through  $\mathbf{D}$  at the expense of multiplication by the scalar  $(-1)^{b_1+b_3+b_5}$ . This is depicted in Figure 26.*

FIGURE 26. Resolving a diagram as in Proposition 4.22 for  $\mathbf{D}$  an invisible  $i$ -diagonal.

*Proof.* We first consider the double crossing with a  $\mathbf{B}_2$  brick (the  $\mathbf{B}_3$  brick case is similar and left an exercise for the reader). First apply relation (1.6) to pull  $A$  through the  $(i-1)$ -ghost to obtain two terms; one with a dot on the  $i$ -strand and one with a dot on the  $(i-1)$ -strand. The diagram with a dot on the  $(i-1)$ -strand is zero by relation (1.4). We apply relations (1.3) and (1.4) to the other diagram and obtain a diagram with a single crossing as depicted in Figure 27.

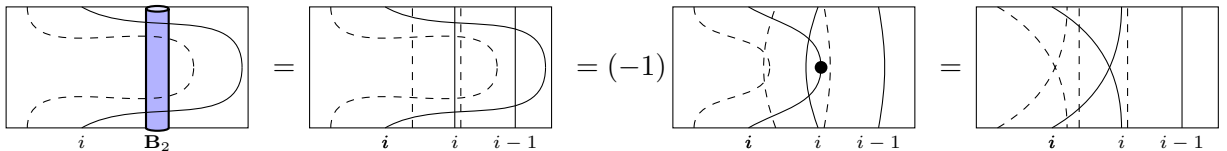


FIGURE 27. The first equality follows by definition. The second equality follows from relation (1.6) where the error term is zero by relation (1.4). The third equality follows from relations (1.3) and (1.4).

Observe that the  $i$ -crossing is attached to the  $i$ -node in the  $\mathbf{B}_2$  (respectively  $\mathbf{B}_3$ ) brick in  $\gamma$ . Repeating arguments from the proof of Proposition 4.17 yields the result.  $\square$

**Proposition 4.23.** *Suppose we have a double crossing of an  $i$ -strand,  $A$ , with a visible  $i$ -diagonal,  $\mathbf{D}$ . We can pull  $A$  through  $\mathbf{D}$  at the expense of scalar multiplication by  $(-1)^{b_1+b_4}$  and acquiring a dot. This is depicted in Figure 28.*

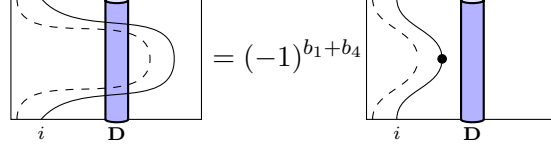


FIGURE 28. Resolving a diagram as in Proposition 4.23 for  $\mathbf{D}$  a visible  $i$ -diagonal.

*Proof.* We claim that we can pull such a strand through a  $\mathbf{B}_1$  at the cost of scalar multiplication by  $(-1)$ . To see this, note that we can pull the  $i$ -strand through the  $(i-1)$ -ghost at the expense of acquiring a dot; we can pull the  $i$ -ghost through the black  $(i+1)$ -strand at the expense of acquiring another dot (both error terms are zero by Lemma 4.16). We thus obtain the diagram on the left-hand side of the equality in Figure 29. Applying relations (1.3) and (1.4) several times, we obtain the diagram in which  $A$  passed through  $\mathbf{B}_1$  at the expense of scalar multiplication by  $(-1)$  (the leftmost diagram after the equality in Figure 29) along with two error terms, which are both zero by Lemma 4.16; see Figure 29.

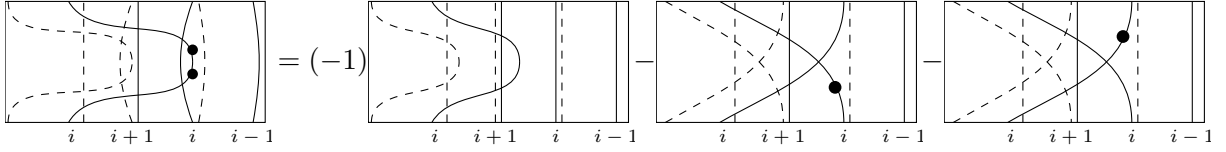


FIGURE 29. The effect of pulling a double crossing  $i$ -strand through a  $\mathbf{B}_1$  brick.

Thus we can pull  $A$  through all the  $\mathbf{B}_1$  bricks at the expense of multiplication by  $(-1)^{b_1}$ . We now wish to consider what happens when we pull  $A$  through a  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick.

We can pull  $A$  through a  $\mathbf{B}_6$  brick at the expense of acquiring a single dot on  $A$  (by relation (1.11)), as required. We can pull  $A$  through a  $\mathbf{B}_5$  or  $\mathbf{B}_4$  brick at the expense of scalar multiplication by  $(-1)^{b_4}$  and an error term in which there is a dot on the  $(i+1)$ -strand (respectively  $(i-1)$ -strand) in the  $\mathbf{B}_5$  (respectively  $\mathbf{B}_4$ ) brick. This error term is zero by Lemma 4.16. We thus obtain the required result.  $\square$

**Proposition 4.24.** *Suppose we have an  $i$ -strand  $A$  (respectively a dotted  $i$ -strand  $A'$ ) next to an invisible (respectively visible)  $i$ -diagonal  $\mathbf{D}$  (respectively  $\mathbf{D}'$ ). We can pull  $A$  (respectively  $A'$ ) through  $\mathbf{D}$  (respectively  $\mathbf{D}'$ ) at the expense of scalar multiplication by  $(-1)^{b_1+b_4}$ . This is depicted in Figure 30.*

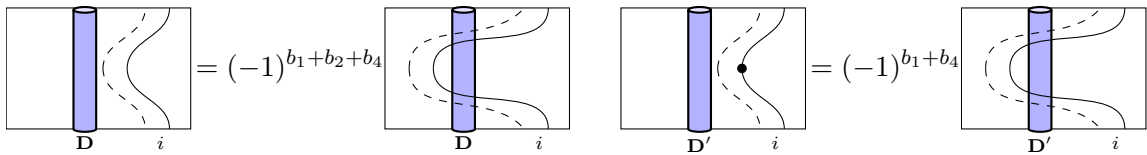


FIGURE 30. Pulling a dotted  $i$ -strand through an invisible  $i$ -diagonal  $\mathbf{D}$  (respectively visible  $i$ -diagonal  $\mathbf{D}'$ ).

*Proof.* We shall first show that  $A$  can pass through a  $\mathbf{B}_2$  brick at the expense of multiplication by  $-1$  (respectively  $+1$ ) and acquiring a dot. The  $\mathbf{B}_3$  case is similar.

We can pull  $A$  through the  $(i-1)$ -ghost in  $\mathbf{B}_2$  and then apply Move 1 to yield two diagrams each with an  $i$ -crossing. In the case that the  $i$ -crossing is bypassed to the left by the  $(i-1)$ -ghost, we can push the crossing to the right and apply Lemma 4.15 to see that this diagram is zero. It remains to consider the diagram with an  $i$ -crossing attached to the  $i$ -node in  $\mathbf{B}_2$  bypassed to the right by the  $(i-1)$ -ghost multiplied by the scalar  $-1$ . By Move 2 we obtain the required result.

We have seen that pulling  $A$  through a  $\mathbf{B}_2$  or  $\mathbf{B}_3$  adds a dot to the strand. Therefore it will suffice to show that the result holds for  $A'$  and  $\mathbf{D}'$  as in the rightmost diagram of Figure 30. First, we show that we can pull  $A'$  through a  $\mathbf{B}_1$  brick at the expense of multiplication by the scalar  $(-1)$ .

By relation (1.7) we can pull  $A'$  through the  $(i-1)$ -ghost at the expense of multiplication by  $-1$  and losing the dot (the error term is zero by Lemma 4.16). We now apply Move 1, followed by Lemma 4.15, followed by Move 2 as above, to obtain the result.

Finally, we can pull the dotted  $A'$  through a  $\mathbf{B}_4$ ,  $\mathbf{B}_5$ , or  $\mathbf{B}_6$  brick using relation (1.7), (1.6) or (1.11) at the expense of multiplication by  $(-1)^{b_4}$  (note that the error term in (1.6) and (1.7) is zero by Lemma 4.16). The result follows.  $\square$

**4.5. The algebra isomorphism for two subquotients each with a single residue.** We now associate a sequence  $\chi(\gamma)$  to the multipartition  $\gamma$ . This sequence records the occurrences of visible and invisible  $i$ -diagonals, and the bricks from which the diagonals are built. We define

$$\chi(\mathbf{D}_x) = (-1)^{b_1} \mathbf{d}_k^j$$

where  $k = 4, 5$ , or  $6$  if the bottom brick of  $\mathbf{D}_x$  is  $\mathbf{B}_4, \mathbf{B}_5$  or  $\mathbf{B}_6$ , respectively and where  $j = 2$  or  $3$  if  $\mathbf{D}_x$  is invisible with top brick  $\mathbf{B}_2$  or  $\mathbf{B}_3$ , respectively, and  $j = 0$  if  $\mathbf{D}_x$  is visible. We then define  $\chi(\gamma)$  to be the sequence  $(\chi(\mathbf{D}_{x_1}), \chi(\mathbf{D}_{x_2}), \dots)$ .

**Example 4.25.** Continuing with Example 4.2, we have that  $\chi(\gamma) = (\mathbf{d}_4^0, \mathbf{d}_4^3, \mathbf{d}_6^0, \mathbf{d}_5^0, \mathbf{d}_5^0)$ .

Let  $-\emptyset$  and  $\emptyset$  denote two formal symbols in what follows.

**Definition 4.26.** We say that two sequences  $\chi$  and  $\bar{\chi}$  are *equivalent*, and write  $\chi \sim \bar{\chi}$ , if one can be obtained from the other by applying the following local identifications within the sequence (i.e. to individual elements or adjacent pairs of elements in the sequence):

- (i)  $(+\mathbf{d}_4^j) = (-\mathbf{d}_5^j)$  and  $(-\mathbf{d}_4^j) = (+\mathbf{d}_5^j)$  for  $j = 0, 2, 3$ ;
- (ii)  $(-\emptyset) = (+\mathbf{d}_k^2, +\mathbf{d}_k^3) = (-\mathbf{d}_k^2, -\mathbf{d}_k^3) = (+\mathbf{d}_k^3, +\mathbf{d}_k^2) = (-\mathbf{d}_k^3, -\mathbf{d}_k^2)$  for  $k = 4, 5$ ;
- (iii)  $(+\mathbf{d}_k^j, -\mathbf{d}_k^j) = (-\mathbf{d}_k^j, +\mathbf{d}_k^j)$  for  $j = 2, 3$  and  $k = 4, 5$ ;
- (iv)  $(\emptyset) = (+\mathbf{d}_6^j, +\mathbf{d}_6^j) = (-\mathbf{d}_6^j, -\mathbf{d}_6^j)$  for  $j = 2, 3$ ;
- (v)  $(-\emptyset, -\emptyset) = (\emptyset)$  and we may delete any  $\emptyset$  from our sequence  $\chi$  without cost.

**Example 4.27.** Let  $e = \bar{e} = 5$ ,  $i = \bar{i} = 0$ ,  $\kappa = \bar{\kappa} = (0)$  and  $\theta = \bar{\theta} = (0)$ . The partition

$$\gamma = (30^6, 28, 20, 19^2, 15, 11, 9, 7, 3^6) \in \mathcal{P}_{326}^1$$

has four addable 0-nodes and

$$\chi(\gamma) = (+\mathbf{d}_4^0, \underbrace{+\mathbf{d}_4^2, +\mathbf{d}_4^3}, \underbrace{+\mathbf{d}_4^3, +\mathbf{d}_4^2}, +\mathbf{d}_4^0, -\mathbf{d}_6^0, -\mathbf{d}_5^3, -\mathbf{d}_5^3, +\mathbf{d}_5^2, +\mathbf{d}_5^0),$$

where we have indicated the pairs of adjacent elements to which we can apply the local identification (ii). The partition

$$\bar{\gamma} = (10^4, 9, 5^4, 3^3, 1^8) \in \mathcal{P}_{86}^1$$

also has four addable 0-nodes and

$$\chi(\bar{\gamma}) = (+\mathbf{d}_4^0, +\mathbf{d}_4^0, -\mathbf{d}_6^0, -\mathbf{d}_5^3, -\mathbf{d}_5^3, +\mathbf{d}_5^2, +\mathbf{d}_5^0).$$

These two sequences are easily seen to be equivalent using (ii) and (v) above.

**Theorem 4.28.** *If  $\gamma$  and  $\bar{\gamma}$  are two multipartitions which are  $i$ - and  $\bar{i}$ -admissible, respectively, with  $\chi(\gamma) \sim \chi(\bar{\gamma})$ , then  $A_\Gamma \cong A_{\bar{\Gamma}}$  as graded  $\mathbb{k}$ -algebras; the isomorphism is given by  $\Phi$  from Proposition 4.11.*

*Proof.* By Proposition 4.11 we need only check that  $\Phi(C_{S,T}C_{U,V}) = \Phi(C_{S,T})\Phi(C_{U,V})$ . We suppose that our product diagram  $D = C_{S,T}C_{U,V}$  contains a neighbourhood in which some  $i$ -strand,  $A$ , passes through an  $i$ -diagonal,  $\mathbf{D}$ . We shall show that resolving the crossing and then applying  $\Phi$ , we obtain the same result as if we resolve the crossing in  $\Phi(D)$ . There are five possible cases that we need to check for each diagonal. The five cases are: (a) moving a dot through a crossing of  $A$  with  $\mathbf{D}$  (b&c) an  $i$ -crossing passes through  $\mathbf{D}$  (d&e) a double crossing of  $A$  with  $\mathbf{D}$ . These are the same cases as those considered in Figure 31.

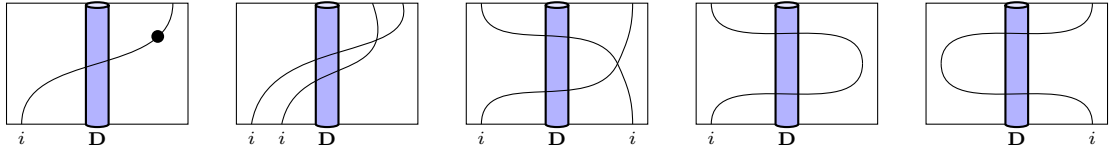


FIGURE 31. The five possible diagrams we need to consider, for  $\mathbf{D}$  an arbitrary  $i$ -diagonal.

First note that cases (a) and (b) are trivial for all identifications (i)–(v) in Definition 4.26, from Proposition 4.21 and the defining relations of  $A(n, \theta, \kappa)$ , respectively.

For the identification (i), we now look at cases (c), (d) and (e). In case (c), the result follows by Proposition 4.17 and the fact that (i) preserves the parity of  $b_1 + b_5$ ; in case (d), the result follows by Propositions 4.22 and 4.24 and the fact that (i) preserves the parity of  $b_1 + b_4$  and  $b_1 + b_3 + b_5$ ; in case (e), the result follows by Proposition 4.23 and the fact that (i) preserves the parity of  $b_1 + b_4$  and  $b_1 + b_2 + b_4$  – these all follow as our equivalence is defined so that these parities are all preserved by  $\Phi$ .

We now address the identifications in (ii). We aim to show that we can resolve a diagram with a pair of (consecutive)  $i$ -diagonals  $\mathbf{D}_{x_1}$  and  $\mathbf{D}_{x_2}$ , with  $\chi(\mathbf{D}_{x_1}) = \pm \mathbf{d}_k^2$  and  $\chi(\mathbf{D}_{x_2}) = \pm \mathbf{d}_k^3$  (or vice versa), for  $k = 4$  or  $5$ , and an  $i$ -crossing as in case (c) without cost or a double crossing as in cases (d&e) at the expense of multiplication by scalar  $-1$ .

First, consider what happens when we resolve an  $i$ -crossing as in case (c). We first pull the  $i$ -crossing through  $\mathbf{D}_{x_2}$ , to obtain an error term in which the crossing is undone, and then through  $\mathbf{D}_{x_1}$  to obtain another error term in which the crossing is undone, applying Proposition 4.17 twice. The resulting sum of three diagrams is depicted in Figure 32 in the case that  $(\chi(\mathbf{D}_{x_1}), \chi(\mathbf{D}_{x_2})) = (\mathbf{d}_4^2, \mathbf{d}_4^3)$ .

Resolving the two error terms using Propositions 4.22 and 4.24, we get the diagram with both  $i$ -strands vertical, with coefficient

$$(\dagger) \quad (-1)^{b_1(x_1)+b_5(x_1)+b_1(x_2)+b_3(x_1)+b_5(x_2)} + (-1)^{b_1(x_1)+b_5(x_1)+b_1(x_2)+b_2(x_2)+b_4(x_2)}$$

where  $b_k(x_j)$  denotes the number of bricks  $\mathbf{B}_k$  in the  $i$ -diagonal  $\mathbf{D}_{x_j}$ . It is simple to check that this coefficient is zero in all cases covered in (ii), and therefore we can pass an  $i$ -crossing through the pair  $(\mathbf{D}_{x_1}, \mathbf{D}_{x_2})$  without cost.

Next, consider what happens when we pull a double  $i$ -crossing as in case (d) (respectively (e)) through through the pair of (consecutive)  $i$ -diagonals  $\mathbf{D}_{x_1}, \mathbf{D}_{x_2}$  with  $\chi(\mathbf{D}_{x_1}) = \pm \mathbf{d}_k^2$  and  $\chi(\mathbf{D}_{x_2}) = \pm \mathbf{d}_k^3$  (or vice versa), for  $k = 4$  or  $5$ .

We can pull the  $i$ -strand through both diagonals, applying Proposition 4.22 (respectively Proposition 4.24) twice, at the expense of multiplication by

$$(\dagger\dagger) \quad (-1)^{b_1(x_1+x_2)+b_3(x_1+x_2)+b_5(x_1+x_2)} \quad \text{or} \quad (-1)^{b_1(x_1+x_2)+b_2(x_1+x_2)+b_4(x_1+x_2)}$$



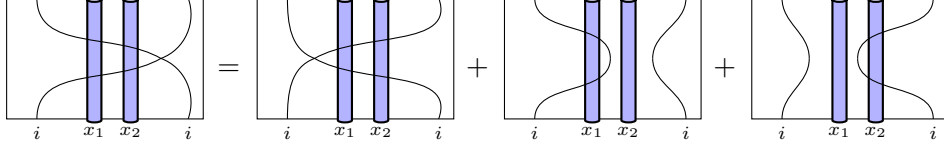


FIGURE 32. The result of pulling an  $i$ -crossing through a pair of  $i$ -diagonals  $\mathbf{D}_{x_1}$  and  $\mathbf{D}_{x_2}$  with  $\chi(\mathbf{D}_{x_1}) = \mathbf{d}_4^2$ ,  $\chi(\mathbf{D}_{x_2}) = \mathbf{d}_4^3$ .

respectively, where  $b_k(x_1 + x_2) := b_k(x_1) + b_k(x_2)$ . In all cases covered in (ii), we have that  $b_1(x_1) = b_1(x_2)$ ,  $b_3(x_1) = b_3(x_2) \pm 1$  (respectively  $b_2(x_1) = b_2(x_2) \pm 1$ ) and  $b_5(x_1) = b_5(x_2)$  (respectively  $b_4(x_1) = b_4(x_2)$ ), and so the scalar is  $-1$  for both (d) and (e), as claimed.

We have shown that we can pull an  $i$ -crossing through any pair of consecutive invisible  $i$ -diagonals as in case (ii) without cost. We have also shown that we can pull a single  $i$ -strand through at the cost of multiplication by  $-1$ ; and therefore  $\Phi$  respects the identifications in (ii). Moreover, we can clearly pull an  $i$ -crossing or a single  $i$ -strand through two such pairs without cost (as  $(-1)^2 = 1$ ), and therefore  $\Phi$  also respects the identifications in (v).

In case (iii) one can argue as above independently of  $k$ , and obtain coefficient  $+2$  or  $-2$  in  $(\dagger)$  for  $j = 3$  or  $j = 2$ , respectively. The coefficients in  $(\dagger\dagger)$  are easily seen to both be  $-1$  for  $j = 2$  or  $3$ . The identifications in (iv) follow similarly to those in (ii), but with scalars in  $(\dagger\dagger)$  both being equal to  $+1$ . The result follows as  $\Phi$  respects these coefficients.  $\square$

**Corollary 4.29.** *Let  $A(n, \theta, \kappa)$  and  $A(\bar{n}, \bar{\theta}, \bar{\kappa})$  be two diagrammatic Cherednik algebras. Let  $\gamma$  be an  $i$ -admissible multipartition and  $\bar{\gamma}$  an  $\bar{i}$ -admissible multipartition, with  $\chi(\gamma) \sim \chi(\bar{\gamma})$ . We have that*

$$d_{\lambda\mu}(t) = d_{\bar{\lambda}\bar{\mu}}(t)$$

for all  $\lambda, \mu \in \Gamma$  and  $\bar{\lambda}, \bar{\mu} \in \bar{\Gamma}$ . We also have that

$$\text{Ext}_{A(n, \theta, \kappa)}^j(\Delta(\lambda), \Delta(\mu)) \cong \text{Ext}_{A(\bar{n}, \bar{\theta}, \bar{\kappa})}^j(\Delta(\bar{\lambda}), \Delta(\bar{\mu}))$$

for all  $\lambda, \mu \in \Gamma$ ,  $\bar{\lambda}, \bar{\mu} \in \bar{\Gamma}$  and  $j \geq 0$ .

*Proof.* The graded decomposition numbers and higher extension groups are preserved under the isomorphisms in the proof of Theorem 4.28 and by (graded analogues of) the results for (co-)saturated idempotent sub- and quotient algebras in [Don98, Appendix].  $\square$

**Theorem 4.30.** *Let  $\gamma$  be an  $i$ -admissible multipartition and suppose that the  $e$ -multicharge  $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$  contains  $i \in \mathbb{Z}/e\mathbb{Z}$  as a constituent with multiplicity 0 or 1. The graded decomposition numbers of  $A(n, \theta, \kappa)$  over  $\mathbb{k}$  can be given in terms of nested sign sequences as follows:*

$$d_{\lambda\mu}(t) = \sum_{\omega \in \Omega(\lambda, \mu)} t^{\|\omega\|}$$

for  $\lambda, \mu \in \Gamma$  such that  $\mu \trianglelefteq_{\theta} \lambda$ .

*Proof.* Suppose that  $\chi(\gamma)$  can be broken into 3 parts (i) a sequence of  $\pm \mathbf{d}_4^j$  for  $j = 0, 2, 3$  (ii) either one or zero  $\pm \mathbf{d}_6^j$  for  $j = 0, 2, 3$  (iii) a sequence of  $\pm \mathbf{d}_5^j$  for  $j = 0, 2, 3$  in order. Then there clearly exists a level 1 partition  $\bar{\gamma}$  such that  $\chi(\gamma) \sim \chi(\bar{\gamma})$  and the result follows from Theorem 4.28 and [TT13, Theorem 4.4].

By assumption,  $\chi(\gamma)$  is a sequence which has a maximum of one entry  $\pm \mathbf{d}_6^j$  for  $j = 0, 2, 3$ . Therefore one can apply the identification (i) of Definition 4.26 to swap all the subscripts of entries  $\mathbf{d}_k^j$  for  $j = 0, 2, 3$ ,  $k = 4, 5$  to obtain some  $\bar{\chi} \sim \chi(\gamma)$  which has the above form. Thus, the result follows.  $\square$

**Example 4.31.** Let  $e = 3$  and  $\kappa = (0)$  and  $i = 2$ . We want to calculate  $d_{\lambda\mu}(t)$  for  $\mu = (10^2, 9^2, 8, 7^2, 6^3, 5, 4^3, 3, 2^3, 1^2)$  and  $\lambda = (10^2, 9^2, 8^2, 7^2, 6, 5^3, 4^2, 3, 2^2, 1^2)$ . By Example 2.6, we have that

$$d_{\lambda\mu}(t) = t^{11} + 2t^9 + 2t^7 + t^5.$$

**Example 4.32.** Let  $e = 3$ ,  $\kappa = (2, 1)$ ,  $i = 0$ , and let  $\theta = (0, 1)$ ,  $g = 2$  (note that  $\theta$  is a FLOTW weighting). The bipartition

$$\gamma = ((7, 5, 3, 1^2), (5^2, 4, 2^2, 1^2))$$

is 0-admissible. We have that  $\chi(\gamma) = (+\mathbf{d}_4^0, +\mathbf{d}_4^0, -\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_5^0, -\mathbf{d}_5^0, -\mathbf{d}_5^0, +\mathbf{d}_5^0, +\mathbf{d}_5^0)$ . Let  $\bar{e} = 4$ ,  $\bar{\kappa} = (1)$ ,  $\bar{\theta} = (0)$  and  $\bar{i} = 0$ . The partition

$$\gamma = (19, 18, 17^3, 16, 13, 12, 11, 8^3, 7, 6, 5, 2^2)$$

is 0-admissible with  $\chi(\gamma) = \chi(\bar{\gamma})$ .

The algebras  $A_\Gamma$  and  $A_{\bar{\Gamma}}$  are isomorphic (as graded  $\mathbf{k}$ -algebras) and the graded decomposition numbers of the latter may be calculated using [TT13, Theorem 4.4] in terms of nested sequences. Given  $\lambda = ((8, 5, 3, 1^3), (6, 5^2, 3, 2, 1^3))$ ,  $\mu = ((7, 5, 4, 2, 1^2), (5^3, 2^3, 1^2)) \in \Gamma$  we leave it as an exercise for the reader to show that the unique well-nested path in this case is the generic latticed path with norm 11. Therefore,

$$d_{((8,5,3,1^3),(6,5^2,3,2,1^3)),((7,5,4,2,1^2),(5^3,2^3,1^2))}(t) = t^{11}$$

for  $A(43, (0, 1), (2, 1))$ .

**Example 4.33.** Continuing with Example 4.27, we let  $\gamma = (30^6, 28, 20, 19^2, 15, 11, 9, 7, 3^6) \in \mathcal{P}_{326}^1$  and  $\bar{\gamma} = (10^4, 9, 5^4, 3^3, 1^8) \in \mathcal{P}_{86}^1$ . By Example 4.27, we have that  $\chi(\gamma) \sim \chi(\bar{\gamma})$ . The partially ordered sets  $\Gamma$  and  $\bar{\Gamma}$  each consist of six partitions and define natural subquotients of the corresponding Schur algebras (as in [Don98]). By [Webb, Corollary 3.11] these algebras are Morita equivalent to the corresponding subquotients  $A_\Gamma$  and  $A_{\bar{\Gamma}}$  of the diagrammatic Cherednik algebras.

Therefore by Theorem 4.28, the subquotients of the classical Schur algebras of degrees 328 and 88 labelled by the sets  $\Gamma$  and  $\bar{\Gamma}$  are Morita equivalent (to each another).

*Remark 4.34.* By Theorem 4.28, one can calculate decomposition numbers for more general  $\kappa$  if one puts restrictions on  $\theta$ ; see Example 4.35, below.

**Example 4.35.** Let  $e = 5$ ,  $\kappa = (0, 0)$ , and  $\theta \in \mathbb{R}^2$  denote a FLOTW weighting. The bipartition  $\gamma = ((10, 8, 7, 5^3, 3^3), (5, 4, 3^5, 2, 1^2))$  is 0-admissible with

$$\chi(\gamma) = (+\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_6^2, +\mathbf{d}_6^2, +\mathbf{d}_5^0, +\mathbf{d}_5^0, +\mathbf{d}_5^0).$$

Now let  $e = 5$ ,  $\kappa = (1)$  and  $\theta = (0)$ . If  $\bar{\gamma} = (14, 12, 11, 9, 8, 5^2, 3, 2, 1^2)$ , then

$$\chi(\bar{\gamma}) = (+\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_4^0, +\mathbf{d}_5^2, +\mathbf{d}_5^0, +\mathbf{d}_5^0)$$

and  $\chi(\gamma) \sim \chi(\bar{\gamma})$ , by identifications  $(iv)$  and  $(v)$  in Definition 4.26. Thus, we can still use Theorem 4.28 to calculate  $d_{\lambda\mu}(t)$  for  $\lambda, \mu \in \Gamma$ . For instance, by Corollary 4.29,

$$d_{((11,8^2,5^3,3^3),(5,4,3^6,1^2)),((10,8,7,5^3,3^3),(6,4,3^6,1^3))}(t) = d_{(15,12^2,9,8,5^2,3^2,1^2),(14,12,11,9^2,5^2,3^2,1^3)}(t),$$

where the decomposition numbers are taken in the relevant algebra. We can now use Theorem 4.30 to find that

$$d_{(15,12^2,9,8,5^2,3^2,1^2),(14,12,11,9^2,5^2,3^2,1^3)}(t) = t^5.$$

## 5. TENSOR PRODUCT FACTORISATION FOR NON-ADJACENT RESIDUES

Throughout this paper, we have constructed isomorphisms between subquotient algebras corresponding to subsets of  $\mathcal{P}_n^l$  consisting of multipartitions which differ by moving nodes of a single, fixed residue. In this section, we generalise these results to subsets of multipartitions which differ by moving nodes of many distinct residues, as long as these residues are nonadjacent. We prove that one can factorise these algebras as a tensor product of the smaller, single residue subalgebras. This lifts results of [CT16] to an isomorphism of algebras which holds over fields

of arbitrary characteristic. In particular, we obtain the graded decomposition numbers of these algebras as products of the graded decomposition numbers of the smaller algebras (over arbitrary fields).

We suppose that  $\mathcal{M}$  is a multiset of residues from an adjacency-free residue set  $S \subset I$  as in Section 3, and let  $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{e-1}$  denote the disjoint decomposition of the multiset  $\mathcal{M}$  into distinct residues; we let  $m_r = |\mathcal{M}_r|$  for  $0 \leq r \leq e-1$ . Note that since  $S$  is adjacency-free, some of these multisets are empty. We let  $\Gamma = \Gamma(\mathcal{M})$  and  $\Gamma_r := \Gamma(\mathcal{M}_r)$  for  $0 \leq r \leq e-1$ .

**Lemma 5.1.** *For an adjacency-free set  $\mathcal{M}$ , we have a bijection*

$$\psi : \Gamma \longrightarrow \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_{e-1}$$

which is given by  $\psi(\lambda) = \psi_0(\lambda) \times \psi_1(\lambda) \times \cdots \times \psi_{e-1}(\lambda)$  where  $\psi_i(\lambda)$  is obtained from  $\lambda$  by deleting all nodes of  $\lambda \setminus \gamma$  whose residue is not equal to  $i \in \mathbb{Z}/e\mathbb{Z}$ .

*Proof.* The adjacency-free condition ensures that no two nodes in  $\lambda \setminus \gamma$  appear in the same row or column for any  $\lambda \in \Gamma$ . The result follows.  $\square$

**Proposition 5.2.** *For an adjacency-free set  $\mathcal{M}$ , we have a bijection*

$$\psi : \text{SStd}(\lambda, \mu) \longrightarrow \prod_{r=0}^{e-1} \text{SStd}(\psi_r(\lambda), \psi_r(\mu))$$

for  $\lambda, \mu \in \Gamma$ . This is given by setting  $\psi(\mathbf{T}) = \psi_0(\mathbf{T}) \times \psi_1(\mathbf{T}) \times \cdots \times \psi_{e-1}(\mathbf{T})$  where  $\psi_i(\mathbf{T})$  is simply obtained by restriction of the domain,  $\psi_i(\mathbf{T}) = \mathbf{T} \downarrow_{\psi_i(\lambda)}$ . This lifts to a graded vector space isomorphism over  $\mathbb{k}$ , given by

$$\Psi : A_\Gamma(\mathcal{M}, \theta) \rightarrow A_{\Gamma_0}(\mathcal{M}_0, \theta) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A_{\Gamma_{e-1}}(\mathcal{M}_{e-1}, \theta)$$

where

$$\Psi(C_{\mathbf{ST}}) = C_{\psi_0(\mathbf{S})\psi_0(\mathbf{T})} \otimes C_{\psi_1(\mathbf{S})\psi_1(\mathbf{T})} \otimes \cdots \otimes C_{\psi_{e-1}(\mathbf{S})\psi_{e-1}(\mathbf{T})}.$$

*Proof.* Given  $\lambda, \mu \in \Gamma_m$ , the set  $\text{SStd}(\lambda, \mu)$  is particularly simple to describe. Namely,  $\text{SStd}(\lambda, \mu)$  consists of the bijective residue-preserving maps such that

- $\mathbf{T}(r, c, k) = \mathbf{i}_{(r, c, k)}$  for  $(r, c, k) \in [\gamma]$ ;
- $\mathbf{T}(r, c, k) = \mathbf{i}_{(r', c', k')}$  for  $(r, c, k) \in [\lambda \setminus \gamma]$ ,  $(r', c', k') \in [\mu \setminus \gamma]$  such that  $\mathbf{i}_{(r', c', k')} \geq \mathbf{i}_{(r, c, k)}$ .

The adjacency-free condition ensures that no two nodes in  $\lambda \setminus \gamma$  (or  $\mu \setminus \gamma$ ) for any  $\lambda, \mu \in \Gamma$  appear in the same row or column. It follows from Definition 1.7 that the map  $\Psi$  is a bijection.

As the bases are in bijection, the map  $\Psi$  is clearly a vector space isomorphism. The non-adjacency condition ensures that for distinct residues  $i, j \in S$ , the crossings of  $i$ -strands (or their ghosts) with  $j$ -strands (or their ghosts) do not provide any non-zero contributions to the grading and so the grading is preserved.  $\square$

**Example 5.3.** Let  $e = 4$ ,  $\gamma = (\emptyset, \dots, \emptyset) \in \mathcal{P}_0^7$ ,  $\kappa = (3, 1, 3, 3, 3, 1, 3)$ ,  $\mathcal{M} = \{1^1, 3^3\}$ ,  $g = 0.99$  and  $\theta = (-3, -1, 1, 3, 5, 9, 11)$ . There are  $2 \times \binom{5}{3} = 20$  simple modules for this algebra. Consider the space  $\Delta_\mu(\lambda)$  for  $\lambda = ((1), (1), \emptyset, (1), (1), \emptyset, \emptyset)$  and  $\mu = (\emptyset, \emptyset, \emptyset, (1), (1), (1), (1))$ . We have that  $\text{SStd}(\lambda, \mu)$  has two elements,  $\mathbf{S}$  and  $\mathbf{T}$ , of degrees 2 and 4, respectively. Therefore  $\text{Dim}_t(\Delta_\mu(\lambda)) = t^4 + t^2$ . The element  $B_{\mathbf{S}}$  of degree 2 is pictured in Figure 33, below.

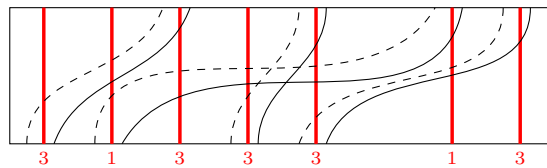


FIGURE 33. The basis element  $B_{\mathbf{S}}$  in Example 5.3.

Under the graded vector space isomorphism in Proposition 5.2,  $B_5$  maps to the tensor product of diagrams depicted in Figure 34.

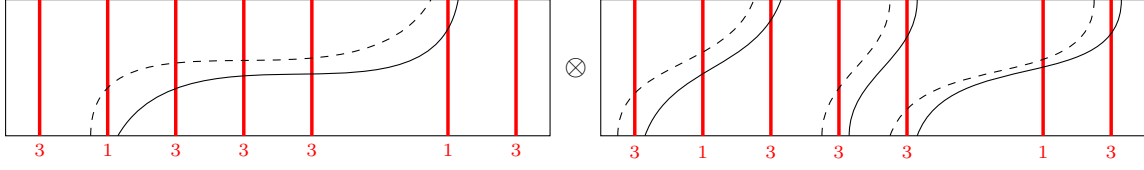


FIGURE 34. Decomposing the diagram in Figure 33 as a tensor product.

**Theorem 5.4.** *If  $\mathcal{M}$  is adjacency-free, then*

$$A_\Gamma(\mathcal{M}, \theta) \cong A_{\Gamma_0}(\mathcal{M}_0, \theta) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A_{\Gamma_{e-1}}(\mathcal{M}_{e-1}, \theta)$$

*as graded  $\mathbb{k}$ -algebras. This isomorphism is given by  $\Psi$  from Proposition 5.2.*

*Proof.* By Proposition 5.2 we need only check that

$$\begin{aligned} \Psi(C_{\text{ST}}C_{\text{UV}}) &= \Psi\left(\sum_{\mathbf{W}, \mathbf{X}} r_{\mathbf{W}, \mathbf{X}} C_{\mathbf{W}, \mathbf{X}}\right) \\ &= \sum_{\mathbf{W}, \mathbf{X}} r_{\mathbf{W}, \mathbf{X}} (C_{\psi_0(\mathbf{W})\psi_0(\mathbf{X})} \otimes \cdots \otimes C_{\psi_{e-1}(\mathbf{W})\psi_{e-1}(\mathbf{X})}) \\ &= \sum_{\mathbf{W}, \mathbf{X}} (r_{\mathbf{W}, \mathbf{X}}^0 C_{\psi_0(\mathbf{W})\psi_0(\mathbf{X})}) \otimes \cdots \otimes (r_{\mathbf{W}, \mathbf{X}}^{e-1} C_{\psi_{e-1}(\mathbf{W})\psi_{e-1}(\mathbf{X})}) \\ &= \Psi(C_{\text{ST}})\Psi(C_{\text{UV}}) \end{aligned}$$

where  $r_{\mathbf{W}, \mathbf{X}} = \prod_{0 \leq i \leq e-1} r_{\mathbf{W}, \mathbf{X}}^i \in \mathbb{k}$  and the sum is over tableaux  $\mathbf{W}, \mathbf{X}$  whose shape and weight are multipartitions belonging to  $\Gamma$ .

Given a strand  $A$  in a diagram  $D$ , we say that the strand is left-justifiable if, upon applying one of the relations (1.1) to (1.15) to a local neighbourhood of  $A$ , we can move the strand  $A$  further to the left. Otherwise, we say that the strand  $A$  is left-justified. In [Webb, Proof of Lemmas 2.5 and 2.20], Webster shows that the process of left-justifying every strand in  $D$  eventually terminates, and that the result is a linear combination of the basis elements from  $\mathcal{C}$ .

Therefore, by left-justifying every strand in the diagram  $C_{\text{ST}}C_{\text{UV}}$  we can rewrite this product as a linear combination of basis elements of  $A_\Gamma(\mathcal{M}, \theta)$ . Similarly, by left-justifying every strand in each of the diagrams  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})}$  for  $0 \leq i \leq e-1$  we shall obtain a linear combination of basis elements of  $A_{\Gamma_0}(\mathcal{M}_0, \theta) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A_{\Gamma_{e-1}}(\mathcal{M}_{e-1}, \theta)$ .

We shall proceed one tensor component at a time. Given a strand,  $A$ , of residue  $i \in \mathbb{Z}/e\mathbb{Z}$  in  $C_{\text{ST}}C_{\text{UV}}$ , there is a corresponding strand,  $\Psi_i(A)$ , of residue  $i \in \mathbb{Z}/e\mathbb{Z}$  in  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})}$  which has the same northern and southern terminating points. We shall say that these strands are paired. Similarly, for any fixed  $0 \leq i \leq e-1$  and any given node of  $\gamma$ , we have corresponding vertical strands in each of the diagrams  $C_{\text{ST}}C_{\text{UV}}$  and  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})}$ . We shall say that these strands are paired, and denote them by  $A$  and  $\Psi_i(A)$  as before. We shall proceed one tensor component at a time. When considering the  $i$ th component, we shall proceed by applying local relations in unison to a neighbourhood of  $A$  and the corresponding neighbourhood of  $\Psi_i(A)$ . Of course, there are strands in  $C_{\text{ST}}C_{\text{UV}}$  (of residue not equal to  $i \in \mathbb{Z}/e\mathbb{Z}$ ) which do not have counterparts in  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})}$ . We shall address this problem separately below.

If  $i \in \mathbb{Z}/e\mathbb{Z}$  is such that  $m_i = 0$ , then we have that  $r_{\mathbf{W}, \mathbf{X}}^i = 1$  if  $\mathbf{T}^\gamma = \psi_i(\mathbf{S}) = \psi_i(\mathbf{T}) = \psi_i(\mathbf{U}) = \psi_i(\mathbf{V}) = \mathbf{W} = \mathbf{X}$ , and 0 otherwise. This is simply because  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})} = 1_\gamma 1_\gamma$ , and all strands in the diagram are already left-justified (in particular  $1_\gamma$  is itself a basis element). Now, pick any  $i \in \mathbb{Z}/e\mathbb{Z}$  such that  $m_i \neq 0$ . Pick a left-justifiable  $i$ -strand,  $A$ , in the diagram  $C_{\text{ST}}C_{\text{UV}}$ , then the paired strand  $\Psi_i(A)$  in the diagram  $C_{\psi_i(\mathbf{S})\psi_i(\mathbf{T})}C_{\psi_i(\mathbf{U})\psi_i(\mathbf{V})}$  is also left-justifiable (because

the only non-trivial relations are between strands of adjacent residues, which are common to both diagrams). We pull these strand to the left in unison. Along the way we shall deal with

- (i) neighbourhoods in which  $A$  encounters a strand in  $C_{\mathcal{S}\tau}C_{\mathcal{U}\nu}$  which is not paired with any strand in  $C_{\psi_i(\mathcal{S})\psi_i(\tau)}C_{\psi_i(\mathcal{U})\psi_i(\nu)}$ ;
- (ii)  $j$ -diagonals of  $\gamma$  which are common to both diagrams for  $j \neq i, i \pm 1$ ;
- (iii)  $i$ -diagonals common to both diagrams.

(Note that the multipartition  $\gamma$  is a union of the nodes in its  $i$ -diagonals and its  $j$ -diagonals for  $j \neq i, i \pm 1$ , so this list is exhaustive.)

In case (i), we note that these strands are all of residue not equal to  $i-1, i$ , or  $i+1$ . Therefore, we apply relations (1.5) and (1.8) to pull  $A$  through the  $j$ -strand of the diagram in  $A_\Gamma(\mathcal{M}, \theta)$  or simply apply relation (1.1) to pull  $\Psi_i(A)$  through the corresponding empty neighbourhood of the diagram in  $A_{\Gamma_i}(\mathcal{M}_i, \theta)$ , without cost.

Now, for (ii) and (iii) it is clear that pulling  $A$  through the neighbourhood of the diagram in  $A_\Gamma(\mathcal{M}, \theta)$  and  $\Psi_i(A)$  through the corresponding neighbourhood of the diagram in  $A_{\Gamma_i}(\mathcal{M}_i, \theta)$  we obtain the same linear combination of diagrams in both cases. This is simply because the diagrams are locally identical! However, we must also note the following extra information,

- in case (ii), we can apply relations (1.5) and (1.8) to pull  $A$  through the  $j$ -diagonal in the diagram in  $A_\Gamma(\mathcal{M}, \theta)$  and  $\Psi_i(A)$  through the corresponding  $j$ -diagonal in the diagram in  $A_{\Gamma_i}(\mathcal{M}_i, \theta)$ , without cost.
- in case (iii), we can apply Propositions 4.17 and 4.21 to 4.24 to pull  $A$  through the  $i$ -diagonal of  $A_\Gamma(\mathcal{M}, \theta)$  (respectively  $\Psi_i(A)$  through the corresponding  $i$ -diagonal in the diagram in  $A_{\Gamma_i}(\mathcal{M}_i, \theta)$ ) to obtain a linear combination of diagrams which differ from the original diagram *only* in the position and decorations of strands which do not correspond to nodes in  $\gamma$ .

In particular, in both cases all the strands labelled by nodes of  $\gamma$  remain exactly as before. The up-shot of this is that we can left-justify every single  $i$ -strand in  $A_\Gamma(\mathcal{M}, \theta)$  and its paired  $i$ -strand in  $A_{\Gamma_i}(\mathcal{M}_i, \theta)$  to obtain identical linear combinations of diagrams *and* we can do this without affecting any strands labelled by nodes of  $\gamma$ . Therefore the strands labelled by nodes of  $\gamma$  continue to be vertical lines in the configuration of a multipartition and so are left-justified. Therefore, the above process terminates with an element

$$\sum_{\mathbf{W}, \mathbf{X}} r_{\mathbf{W}, \mathbf{X}}^i C_{\psi_i(\mathbf{W})\psi_i(\mathbf{X})} \in A_{\Gamma_i}(\mathcal{M}_i, \theta)$$

and a corresponding element of  $A_\Gamma(\mathcal{M}, \theta)$  (with the same coefficients, but with diagrams which are not yet basis elements, as we must still consider the other residues  $j \neq i$  for  $j \in \mathbb{Z}/e\mathbb{Z}$ ). We remark that at this point we know only that  $\mathbf{W}, \mathbf{X}$  must exist (simply because we can rewrite any product as a linear combination of basis elements) but we have only determined the semistandard tableaux  $\psi_i(\mathbf{W})$  and  $\psi_i(\mathbf{X})$ .

Continuing in this fashion through all the residues  $i \in \mathbb{Z}/e\mathbb{Z}$  we obtain,

$$\Psi\left(\sum_{\mathbf{W}, \mathbf{X}} (r_{\mathbf{W}, \mathbf{X}}^0 \times \cdots \times r_{\mathbf{W}, \mathbf{X}}^{e-1}) C_{\mathbf{W}, \mathbf{X}}\right) = \sum_{\mathbf{W}, \mathbf{X}} (r_{\mathbf{W}, \mathbf{X}}^0 C_{\psi_0(\mathbf{W})\psi_0(\mathbf{X})}) \otimes \cdots \otimes (r_{\mathbf{W}, \mathbf{X}}^{e-1} C_{\psi_{e-1}(\mathbf{W})\psi_{e-1}(\mathbf{X})})$$

and setting  $r_{\mathbf{W}, \mathbf{X}} = r_{\mathbf{W}, \mathbf{X}}^0 \times \cdots \times r_{\mathbf{W}, \mathbf{X}}^{e-1}$  we obtain the required result.  $\square$

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